

Character expansion for HOMFLY polynomials. II. Fundamental representation. Up to five strands in braid

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ABSTRACT

Character expansion is introduced and explicitly constructed for the (non-colored) HOMFLY polynomials of the simplest knots. Expansion coefficients are not the knot invariants and can depend on the choice of the braid realization. However, the method provides the simplest systematic way to construct HOMFLY polynomials directly in terms of the variable $A = q^N$: a much better way than the standard approach making use of the skein relations. Moreover, representation theory of the simplest quantum group $SU_q(2)$ is sufficient to get the answers for all braids with $m < 5$ strands. Most important we reveal a hidden hierarchical structure of expansion coefficients, what allows one to express all of them through extremely simple elementary constituents. Generalizations to arbitrary knots and arbitrary representations is straightforward.

1 Introduction

Character expansions play an increasing role in modern studies of exact (non-perturbative) partition functions in various quantum field theory models, from QCD [1] to conformal field theories [2] and matrix models [3]. They help to reveal explicit and, especially, hidden symmetries, in particular, are important for the study of hidden integrability properties [4], which reflect the existence of non-linear relations between the correlation functions. Not surprisingly, these expansions provide also a powerful tool for explicit calculations.

So far, most applications of character expansions arise in Yang-Mills theories, especially on the lattice [5], and in the theory of matrix models [6]. In [7] we suggested to extend these considerations to Chern-Simons theory of knots [8], namely to the theory of HOMFLY polynomials [9] and superpolynomials [10], the key players in the theory of knot invariants. As explained in [7], in the well studied case of the torus knots, character expansions are indeed useful to explicitly express the linear and non-linear relations between the HOMFLY polynomials in the form of the "A-polynomial" difference equations and the Hirota/Plucker relations respectively. This adds to the previous demonstration in [11] of how the character decompositions can be used to construct generic superpolynomials and superseries for toric knots and links, which is by itself a highly non-trivial problem. Though all these results do not extend literally to arbitrary knots, they provide a serious motivation for the study of character expansions in knot theory. Perhaps surprisingly, this direction did not attract as much attention so far as it clearly deserves. The lucky exceptions are by-now classical papers [12], but they do not go too deep into the structure of the expansions and thus do not reveal clearly its very interesting properties.

Perhaps, the reason for an insufficient attention to character expansions in the case of knots, is that they are *not* knot invariants: the expansion depends on the braid realization of the knot, and knot

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equivalent braid realizations provide different expansions of one and the same HOMFLY polynomial. However, we shall see that even for knot invariants this technique is extremely useful: it provides very simple formulas directly for the HOMFLY polynomials (i.e. directly in terms of $A = q^N$ rather than for particular values of N), moreover, for entire infinite series of knots (of which the torus series is looking just a non-specific example), what is hardly achievable for the alternative approach based on (computer) application of the skein relations. In fact, there are much more applications of the character expansion [7]: from associating integrable structures with knots to effective dealing with A-polynomials [13].

In the present paper we study the character decomposition of the HOMFLY polynomials [7],¹

$$H_R^K = \sum_{Q \vdash m|R} h_R^Q S_R^* \quad (1)$$

where S_R^* are the Schur functions (characters of the linear groups $GL(N)$ taken at the special point of the time-variable space $p_k = p_k^*$, see eq.(7) below, and find expressions for the coefficients h_R^Q in the Turaev-Reshetikhin formalism [14], i.e. in terms of the products of quantum \mathcal{R} -matrices along the m -strand braid, arising in a $2d$ projection of the knot \mathcal{K} (from the point of view of Chern-Simons theory this corresponds to evaluating the functional integral in the temporal gauge $\mathcal{A}^0 = 0$, see [15] for the current status of such an interpretation). As already mentioned, such a decomposition, and particular coefficients h_R^Q depend on the braid realization: it is enough to mention that even the number of strands m is not by itself a knot invariant. We shall see, however, that the dependence on the choice of the braid realization is not as strong as it could be, perhaps even some knot covariance of the coefficient sets $\{h_R^Q\}$ can be found in this context to substitute/generalize the knot invariance of the HOMFLY polynomials H_R^K .

The next step is to switch from the ordinary to Tanaka-Krein representation of the \mathfrak{R} -matrices (for a categorical approach to the construction described here see [16]). Namely, with the m -strand braid one naturally associates a decomposition of the representation product,

$$R^{\otimes m} = \oplus_{Q \vdash m|R} \mathcal{M}_Q \otimes Q \quad (2)$$

and the crucial property of \mathfrak{R} -matrices is that they act as units in all the constituent representations Q . Thus, the \mathfrak{R} -matrices are naturally projected to the "space of representations"

$$\mathcal{M} = \oplus_{Q \vdash m|R} \mathcal{M}_Q \quad (3)$$

and have there a block-diagonal form

$$\widehat{\mathfrak{R}} = \text{diag}\{\widehat{\mathfrak{R}}^Q\} \quad (4)$$

The matrices $\widehat{\mathfrak{R}}^Q$ can also be diagonalized, but now one should recall that there are actually $m - 1$ different \mathfrak{R} -matrices in the braid realization, with \mathfrak{R}_μ acting at the intersection of the adjacent strands μ and $\mu + 1$. So, each $\widehat{\mathfrak{R}}_\mu^Q$ can be diagonalized, but not for all values of $\mu \in \{1, \dots, m - 1\}$ at once. Instead, $\widehat{\mathfrak{R}}_\mu$ -matrices for different μ are related by the conjugation,

$$\widehat{\mathfrak{R}}_\mu = \widehat{\mathcal{U}}_{\mu\nu} \widehat{\mathfrak{R}}_\nu \widehat{\mathcal{U}}_{\mu\nu}^{-1} \quad (5)$$

where the "mixing" matrices $\widehat{\mathcal{U}}$ can be chosen orthogonal and also have a block diagonal form

$$\widehat{\mathcal{U}}_{\mu\nu} = \text{diag}\{\widehat{\mathcal{U}}_{\mu\nu}^Q\} \quad (6)$$

However, this is not the end of the story. Mixing matrices can be further decomposed into elementary constituents, which appear to exhibit additional universality properties and are directly related

¹As usual, $\sum_{Q \vdash K}$ means a sum over all Young diagrams Q of the size (the number of boxes) equal to K . In what follows, we parameterize Young diagrams by a partition $Q = \{q_1 \geq q_2 \geq \dots \geq 0\}$.

to the Racah coefficients. It looks plausible that, after some work, the coefficients h_R^Q for generic knots can be all expressed in an absolutely explicit form. In this particular paper we demonstrate how all this works in the simplest, still non-trivial case of the fundamental representation $R = [1]$ and the small number of strands $m \leq 5$. This is actually enough to explicitly express *all* the HOMFLY polynomials for all the knots P_i from the Rolfsen table at [17] with $P \leq 8$ through just two discrete functions \varkappa_Q and $C_k = 1/[k]_q$. In further papers of the series this result will be extended to other representations R (to the colored HOMFLY polynomials) and to the broader braids with $m \geq 5$.

Now we explain this general scheme more concretely. First of all, as we already mentioned, the character expansion (1) is taken not an *arbitrary* point in the space of time-variables: it is constrained to just a 2-dimensional slice which encodes all the dependence on the group in the A -dependence,

$$p_k^* = \frac{A^k - A^{-k}}{q - q^{-1}} = \frac{\{A^k\}}{\{q\}} \quad (7)$$

Hereafter, we introduced a useful notation $\{x\} = x - x^{-1}$ to simplify the formulas. For $A = q^N$ these $p_k^* = [N]_q$, where the q -number is defined $[k]_q \equiv (q^k - q^{-k})/(q - q^{-1})$.

The manifest expressions for the Schur functions $S_Q\{p^*\}$ in these special points (7) are quite simple and generalize the standard hook formula [18]:

$$S_Q\{p^*\} = \prod_{(i,j) \in Q} \frac{\{Aq^{i-j}\}}{\{q^{h_{i,j}}\}} \xrightarrow{A=q^N} \prod_{(i,j) \in Q} \frac{[N+i-j]_q}{[h_{i,j}]_q} \quad (8)$$

where $h_{i,j}$ is the hook length.

Now, there are two important facts:

- (i) The general expression for the HOMFLY polynomial within the Reshetikhin-Turaev approach is given

$$H_R^K\{p^*\} = \text{Tr}_{R^{\otimes m}} \mathcal{B}^K = \text{Tr}_{R^{\otimes m}} \prod_s \mathfrak{R}_{\mu(s)}^\pm \quad (9)$$

as a weighted A -dependent trace of an element \mathcal{B}^K of an m -strand braid group, which is a product of quantum (q -dependent) \mathfrak{R} -matrices appearing in the braid in a certain sequence labeled by the index s in (9). Trace here is a weighted trace, see eq.(10) below.

- (ii) As we discussed above, the quantum \mathfrak{R} -matrix acts as a c -number in irreducible representations Q in the decomposition (2). This statement, however, requires a more careful formulation. In (2) we denote by Q the highest weight representation (in practice, it is labeled by the Young diagram) so that the sum runs over highest weights, or Young diagrams. \mathcal{M}_Q is actually an intertwining operator. When the representation Q with the same highest weight appears several times in the expansion of $R^{\otimes m}$, the space of intertwining operators has a non-unit dimension $N_{R^m}^Q = \dim \mathcal{M}_{R^m}^Q$ known as Littlewood-Richardson coefficient. In the present paper R is actually $R = [1]$, and in what follows we often omit the subscript R^m .

These two well-known facts immediately lead to decomposition (1), with coefficients h_R^Q expressed through the eigenvalues of \mathfrak{R} -matrix and the Racah coefficients, which seem to be quite comprehensible. Moreover, the only source of A (and thus N) dependence is the traces

$$\text{Tr}_Q I \equiv \text{ordinary trace}_Q(q^\rho)^{\otimes m} = S_Q^* = S_Q\{p^*\} \quad (10)$$

where we manifestly included the factor q^ρ into the definition of trace. Therefore, the coefficients h_R^Q can be calculated for the smallest possible group $SU(N = l(Q))$, where $l(Q)$ is the number of columns in the Young diagram describing the highest weight of Q . From now on, we denote through Q both the representation and the corresponding Young diagram, hopefully this would not cause any misinterpretation.

Our main goal in this text is a review of the statements (i) and (ii) and their not-so-trivial relation to the usual *straightforward* approaches to evaluation of the HOMFLY polynomials, which do *not* reveal the hidden structure (1). The key point will be reformulation of the braid traces: from matrices in the representation space $R^{\otimes m}$ we switch to those in the space $\mathcal{M} = \oplus_Q \mathcal{M}^Q$ of the intertwining operators, appearing in the decomposition (2). The true meaning of (ii) is that each \mathcal{M}_Q is preserved by action of the \mathfrak{R} -matrices and these latter can be converted into the block-diagonal matrices which act on \mathcal{M} . In what follows we denote \mathfrak{R} -matrices acting on \mathcal{M} by the additional hat, $\hat{\mathfrak{R}}$, and by double brackets, to distinguish them from the ordinary \mathfrak{R} -matrices, acting in the representation spaces $R^{\otimes m}$.

In these terms, the statement (ii) can be given a very explicit form:

$$\mathfrak{R}_\mu = \oplus_Q \hat{\mathfrak{R}}_\mu^Q \otimes I_Q \quad (11)$$

i.e. the \mathfrak{R} -matrices act on the product of representations (2) as

$$\mathfrak{R}_\mu(R^{\otimes m}) = \oplus_Q \hat{\mathfrak{R}}_\mu^Q(\mathcal{M}^Q) \otimes Q \quad (12)$$

This follows from the elementary \mathfrak{R} -matrix relation with $m = 2$,

$$\mathfrak{R}(R_1 \otimes R_2) = \oplus_S \hat{\mathfrak{R}}(\mathcal{M}_{R_1 R_2}^S) \otimes S \quad (13)$$

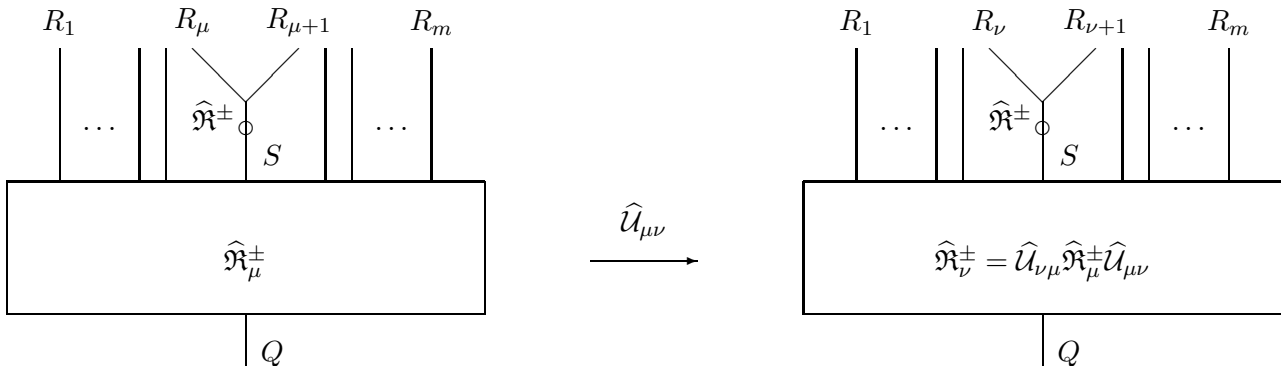
and commutativity of \mathfrak{R} -matrix and comultiplication,

$$\mathfrak{R}\Delta(g)\mathfrak{R}^{-1} = \Delta(g) \quad (14)$$

These $\hat{\mathfrak{R}}$ -matrices can be further diagonalized *within* the \mathcal{M}_Q spaces as well, but not all at once. Each $\hat{\mathfrak{R}}_\mu$ can be diagonalized within \mathcal{M}_Q , but the corresponding basis depends on μ . Basics with different μ are linearly related by $N_{R^m}^Q \times N_{R^m}^Q$ matrices $\hat{\mathcal{U}}_{\mu\nu}^Q = (\hat{\mathcal{U}}_{\nu\mu}^Q)^{-1}$. In other words, all $\hat{\mathfrak{R}}$ -matrices can be expressed through, say, the first one, $\hat{\mathfrak{R}}_\mu = \hat{\mathcal{U}}_{\mu 1} \hat{\mathfrak{R}}_1 \hat{\mathcal{U}}_{1\mu}$, and

$$H_R = \sum_Q S_Q^* \text{Tr}_{\mathcal{M}^Q} \left(\prod_s \hat{\mathcal{U}}_{\mu(s)1} \hat{\mathfrak{R}}_1^\pm \hat{\mathcal{U}}_{1\mu(s)} \right) \quad \text{i.e.} \quad h_R^Q = \text{Tr}_{\mathcal{M}^Q} \left(\prod_s \hat{\mathcal{U}}_{\mu(s)1} \hat{\mathfrak{R}}_1^\pm \hat{\mathcal{U}}_{1\mu(s)} \right) \quad (15)$$

where $\hat{\mathfrak{R}}_1$ can be taken in the diagonal form.



In the main part of the paper we are going to explain the meaning and practical work with formula (15). We explicitly construct the mixing matrices $\hat{\mathcal{U}}_{1\mu}^Q$ for $R = [1]$ and $m < 5$, what allows one to express the known HOMFLY polynomials for all the knots with up to 8 crossings through one and the same set of $\hat{\mathcal{U}}$ -matrices. Moreover, we construct the $\hat{\mathcal{U}}$ -matrices in a special basis, where they can be represented as ordered products of some elementary mixing matrices, which in the case of $[R] = 1$ are essentially 2×2 . The basis is labeled by different fusions of the representations R , i.e. by the decorated trees inside the boxes in the picture, and the elementary constituents of the $\hat{\mathcal{U}}$ -matrices

correspond to flipping of just one edge of the graph (therefore, their matrix elements are given by the Racah coefficients, or fusion matrices).

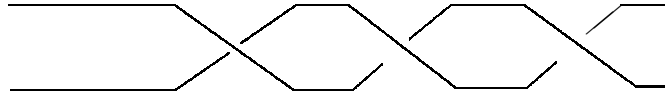
The restrictions to $[R] = 1$ and $m < 5$ are technical: in this case only the Young diagrams Q with no more than two columns or two rows are contributing, and, given the N -independence (universality) of (15), all h_R^Q can be derived from the representation theory of $SU_q(2)$ algebra. When Q with $l(Q)$ columns or rows appear, one needs to extend consideration to $SU_q(l(Q))$ at least. This is straightforward, but in order to avoid overloading the text, we postpone consideration of such examples to separate publications.

The explicit formulae include only group theory data, but of two types: some is known in general for arbitrary Young diagrams (these are eigenvalues r_Q of the quantum R -matrix and values of the Schur functions S_Q^* , eq.(8)), while some do not yet possess such exhaustive expressions (these are the Littlewood-Richardson coefficients and the Racah coefficients).

In the paper, we list the examples in the order of increasing strand number m . The 2-strand case is sensitive only to the Littlewood-Richardson coefficients. The Racah coefficients appear only for 3 and more strands. We interrupt listing the examples in sect.4 with explaining the procedure general construction of the mixing matrix U . After that we give the results for 4 and 5 strands. In fact, for $m = 5$ and $R = [1]$ there is one representation, $Q = [311]$ not obtained by $SU_q(2)$ consideration. In the present paper we restore the corresponding coefficient from the known answers for the HOMFLY polynomials. Some summary and comments are contained in the last section.

2 The 2-strand braids

We begin with the simplest case of the two-strand braid, $m = 2$:



In this case we get only the torus knots and links of the type $[2, n]$, and

$$H_R^{[2,n]} = \text{Tr}_{R \otimes R} \mathfrak{R}^n \quad (16)$$

with integer n . Knots arise for n odd, links for n even. In the latter case, one can also consider

$$H_{R_1, R_2}^{[2,n]} = \text{Tr}_{R_1 \otimes R_2} \mathfrak{R}^n \quad (17)$$

with $R_1 \neq R_2$. For $R_1 \otimes R_2 = \sum_Q N_{R_1 R_2}^Q Q$ one gets

$$H_{R_1, R_2}^{[2,n]} = \sum_Q N_{R_1 R_2}^Q \text{Tr}_Q \mathfrak{R}^n = \sum_Q N_{R_1 R_2}^Q S_Q^* r_Q^n \quad (18)$$

where r_Q is the eigenvalue of \mathfrak{R} in the representation Q . This is the final answer.

In fact, for a single \mathfrak{R} -matrix the eigenvalues r_Q are known in full generality²:

$$r_Q = q^{\varkappa_Q} \quad (19)$$

where \varkappa_R is the eigenvalue of the simplest non-trivial cut-and-join operator $\hat{W}_{[2]}$ [19] on the character eigenfunction $S_R\{p\}$:

$$\text{for } Q = \{q_1 \geq q_2 \geq \dots \geq 0\} \quad \varkappa_Q = \frac{1}{2} \sum_i q_i(q_i + 1 - 2i) = \nu_{Q'} - \nu_Q, \quad \nu_Q = \sum_i (i-1)q_i \quad (20)$$

²Note that we chose in [11] the opposite sign of \varkappa_Q . In the case of HOMFLY polynomials, this sign is inessential to the symmetry of \mathfrak{R} -matrix w.r.t. the replace $q \leftrightarrow q^{-1}$. The symmetry, however, is more complicated in the case of superpolynomials, and one has to choose the sign as in [11] in that case.

and Q' is the transposed Young diagram.

In the simplest case of $R_1 = R_2 = [1]$ one has $[1] \otimes [1] = [2] + [11]$ and the two eigenvalues are just $r_{[2]} = q$ and $r_{[11]} = -\frac{1}{q}$. Thus, (18) in this case turns into

$$H_{[1]}^{[2,n]} = \text{Tr}_{[1] \otimes 2} \mathfrak{R}^n = \text{Tr}_{[2]} \mathfrak{R}^n + \text{Tr}_{[11]} \mathfrak{R}^n = q^n \text{Tr}_{[2]} I + \left(-\frac{1}{q}\right)^n \text{Tr}_{[11]} I = q^n S_2^* + \left(-\frac{1}{q}\right)^n S_{11}^* \quad (21)$$

Of the knots with no more than 8 crossings, the following ones get into the set of the 2-strand knots:

$$3_1 = [2, 3], \quad 5_1 = [2, 5], \quad 7_1 = [2, 7], \quad \dots \quad (22)$$

The first notation refers, e.g., to the Rolfsen Knot Table, see [17], the second one is the standard (dual) notation for the torus knots. For these knots eq.(21) gives:

knot	n	S_2^*	S_{11}^*	A	A^{-1}
3_1	3	q^3	$-q^{-3}$	$q^2 + q^{-2}$	-1
5_1	5	q^5	$-q^{-5}$	$q^4 + 1 + q^{-4}$	$-q^2 - q^{-2}$
7_1	7	q^7	$-q^{-7}$	$q^6 + q^2 + q^{-2} + q^{-6}$	$-q^4 - 1 - q^{-4}$
\dots					

The two columns in the middle contain the two coefficients $h_{[1]}^{[2]}$ and $h_{[1]}^{[11]}$ which stand in front of S_2^* and S_{11}^* correspondingly. In this case, they are especially simple and given by the general formula (21). The last two columns contain the coefficients in front of powers of A in the ratio $H_{[1]}(A)/S_{[1]}^*$, which arise from substitution of $S_{[2]}^*/S_{[1]}^*$ and $S_{[11]}^*/S_{[1]}^*$ into (21). Note that, despite

$$S_1^* = \frac{A - A^{-1}}{q - q^{-1}}, \quad \frac{S_2^*}{S_1^*} = \frac{qA - q^{-1}A^{-1}}{q^2 - q^{-2}}, \quad \frac{S_{11}^*}{S_1^*} = \frac{q^{-1}A - qA^{-1}}{q^2 - q^{-2}} \quad (23)$$

contain non-trivial denominators, they disappear from the HOMFLY polynomials. Of course, the arising explicit expressions for the HOMFLY polynomials coincide with the known expressions from [17] (where $z = q - q^{-1}$, $a = 1/A$, and one has additionally to divide our expressions by the normalization factor A^n).

2.1 Colored HOMFLY polynomials

In fact, the coefficients h_R^Q are known explicitly in far more generality, that is, for all torus knots $[m, n]$ [12, 20]. In the case of $m = 2$ this allows one to extend (21) to arbitrary representations R (i.e. to the

colored HOMFLY polynomials). For example,

$$\begin{aligned}
H_{[2]}^{[2,n]} &= \text{Tr}_{[2] \otimes 2} \mathfrak{R}^n = q^{6n} S_4^* - q^{2n} S_{31}^* + S_{22}^* \\
H_{[11]}^{[2,n]} &= \text{Tr}_{[11] \otimes 2} \mathfrak{R}^n = S_{22}^* - q^{-2n} S_{211}^* + q^{-6n} S_{1111}^* \\
H_{[3]}^{[2,n]} &= \text{Tr}_{[3] \otimes 2} \mathfrak{R}^n = q^{15n} S_6^* - q^{9n} S_{51}^* + q^{5n} S_{42}^* - q^{3n} S_{33}^* \\
H_{[21]}^{[2,n]} &= \text{Tr}_{[21] \otimes 2} \mathfrak{R}^n = q^{5n} S_{42}^* - q^{3n} S_{411}^* - q^{3n} S_{33}^* + q^{-3n} S_{3111}^* + q^{-3n} S_{222}^* - q^{-5n} S_{2211}^* \\
H_{[111]}^{[2,n]} &= \text{Tr}_{[111] \otimes 2} \mathfrak{R}^n = q^{-3n} S_{222}^* - q^{-5n} S_{2211}^* + q^{-9n} S_{21111}^* - q^{-15n} S_{111111}^* \\
&\dots
\end{aligned} \tag{24}$$

2.2 2-strand superpolynomials

According to [11], the switch from HOMFLY to superpolynomials in the case of toric knots is straightforward. In the 2-strand case it is enough to substitute the two \mathfrak{R} -matrix eigenvalues

$$q, -\frac{1}{q} \longrightarrow q, -\frac{1}{t} \tag{25}$$

and the two quantum dimensions

$$\frac{S_2^*}{S_1^*} \longrightarrow \frac{M_2^*}{M_1^*} = \frac{\{Aq\}}{\{qt\}}, \quad \frac{S_{11}^*}{S_1^*} \longrightarrow \frac{1-t^4}{1-q^2t^2} \frac{M_{11}^*}{M_1^*} = \frac{\{At^{-1}\}}{\{t^2\}} \tag{26}$$

The result is

$$P_{[1]}^{[2,2k+1]} = q^{2k+1} M_2^* - \frac{1-t^4}{1-q^2t^2} \frac{q}{t} t^{-2k-1} M_{11}^* = -\frac{M_1^*}{A} \frac{q}{t^{2k+1}} \left[\frac{1-(q^2t^2)^{k+1}}{1-q^2t^2} (-A^2) + t^2 \frac{1-(q^2t^2)^k}{1-q^2t^2} \right] \tag{27}$$

All the coefficients in the emerging polynomial (in the square brackets) are positive integers which are related to dimensions of the Khovanov-Rozhansky homologies [21, 22].

3 The 3-strand braids

This is the first case where there are at least two different \mathfrak{R} -matrices, \mathfrak{R}_1 and \mathfrak{R}_2 , and a mixing matrix U_{12} arise for the first time. On the other hand, since the decomposition

$$[1]^3 = [3] + 2[21] + [111] \tag{28}$$

contains exactly the same number of different representations (three) as there are different powers of A^2 in S^* at the third level, the character expansion of the HOMFLY polynomials in this case is defined unambiguously and can be checked in an independent way.

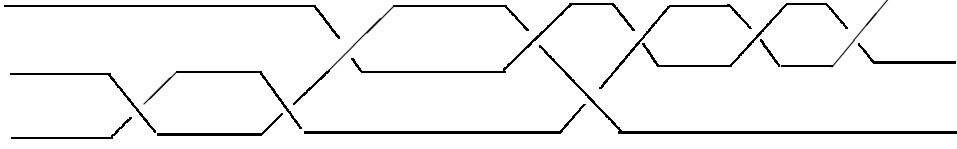
Since there are $N_{12}^{[21]} = 2$ representations [21] in the decomposition (28), the U -matrix will be non-trivial only in this sector, $Q = [21]$, and it will be a 2×2 orthogonal matrix,

$$U = \left(\begin{pmatrix} C & S \\ -S & C \end{pmatrix} \right), \quad \text{while} \quad \mathfrak{R}_1 = \mathfrak{R} \otimes I \tag{29}$$

in this sector is just $\mathfrak{R}_1 = \left(\begin{pmatrix} q & 0 \\ 0 & -\frac{1}{q} \end{pmatrix} \right)$. We remind that the \mathfrak{R} -matrices in the \mathcal{M} space are denoted by the double brackets, in order to distinguish them from the ordinary \mathfrak{R} -matrices, acting in the space

of $[1]^3$. The ordinary \mathfrak{R} matrices depend on N (are of the size $N^3 \times N^3$, while the double bracket (or hatted) ones are 1×1 , 2×2 and 1×1 in the sectors $Q = [3]$, $Q = [21]$ and $Q = [111]$ respectively.

An arbitrary 3-strand braid is parameterized by a sequence of integers $a_1, b_1, a_2, b_2, \dots$ (in this figure $a_1 = -2$, $b_1 = 2$, $a_2 = -1$, $b_2 = 3$: this is knot 8_{10}):



The corresponding $H_{[1]}$ is given by

$$\begin{aligned}
H_{[1]} &= \text{Tr} \left\{ \left(\mathfrak{R} \otimes I \right)^{a_1} \left(I \otimes \mathfrak{R} \right)^{b_1} \left(\mathfrak{R} \otimes I \right)^{a_2} \left(I \otimes \mathfrak{R} \right)^{b_2} \dots \right\} \\
&= \sum_{Q=[111],[21],[3]} \text{tr} \left\{ \left(\hat{\mathfrak{R}}_1^Q \right)^{a_1} \left(\hat{\mathfrak{R}}_2^Q \right)^{b_1} \left(\hat{\mathfrak{R}}_1^Q \right)^{a_2} \left(\hat{\mathfrak{R}}_2^Q \right)^{b_2} \dots \right\} = \\
&= q^{a_1+b_1+a_2+b_2+\dots} S_3^* + \left(-\frac{1}{q} \right)^{a_1+b_1+a_2+b_2+\dots} S_{111}^* + \\
&+ \text{tr} \left\{ \left(\begin{pmatrix} q & 0 \\ 0 & -\frac{1}{q} \end{pmatrix} \right)^{a_1} \underbrace{\left(\begin{pmatrix} C & S \\ -S & C \end{pmatrix} \right) \left(\begin{pmatrix} q & 0 \\ 0 & -\frac{1}{q} \end{pmatrix} \right)^{b_1} \left(\begin{pmatrix} C & -S \\ S & C \end{pmatrix} \right)}_{\dots} \times \\
&\times \left(\begin{pmatrix} q & 0 \\ 0 & -\frac{1}{q} \end{pmatrix} \right)^{a_2} \underbrace{\left(\begin{pmatrix} C & S \\ -S & C \end{pmatrix} \right) \left(\begin{pmatrix} q & 0 \\ 0 & -\frac{1}{q} \end{pmatrix} \right)^{b_2} \left(\begin{pmatrix} C & -S \\ S & C \end{pmatrix} \right)}_{\dots} \dots \right\} S_{21}^*
\end{aligned} \tag{30}$$

We took the orthogonality of U into account. We evaluate C and S explicitly in s.4 below, it turns out that

$$C = \frac{1}{[2]_q} = \frac{1}{q + q^{-1}}, \quad S = \frac{\sqrt{[3]_q}}{[2]_q} = \frac{\sqrt{q^2 + 1 + q^{-2}}}{q + q^{-1}} \tag{31}$$

In the classical limit of $q \rightarrow 1$ this orthogonal matrix U describes rotating at angle $\pi/3$, while at generic q the rotation angle θ ($C = \cos \theta$, $S = \sin \theta$) is not that nice. **Note that all the 3-strand knots are described by a single formula with just two non-trivial entries (29).** All the corresponding HOMFLY polynomials from [17] are certainly immediately reproduced (we remind that in [17] $z = q - q^{-1}$ and $a = 1/A$), see the Table at the next page.

Comment on the Table. 3_1 is a torus knot $[2, 3]$, therefore, it has a natural minimal braid realization as a 2-strand braid, and in this quality it was already considered in the previous section. However, since $[2, 3] = [3, 2]$, it also possesses a 3-strand representation and, hence, is also present in the Table. The corresponding $\mathcal{H}_R\{p\} \equiv \sum_R h_R^Q S_Q(p)$ are, of course, different:

$$\mathcal{H}_R^{[3,2]}(p) \neq \mathcal{H}_R^{[2,3]}(p) \tag{32}$$

i.e. $\mathcal{H}_R(p)$'s are *not* knot invariants, but their restrictions on the subspace *do* coincide:

$$H_R^{[3,2]} = \mathcal{H}_R^{[3,2]} \{p^*\} = \mathcal{H}_R^{[2,3]} \{p^*\} = H_R^{[2,3]} \tag{33}$$

and this is indeed seen in the last columns of the Tables of this and the previous sections (for $R = [1]$).

8_{19} is a torus knot [3, 4], therefore, it possesses also another braid representation, but this time it is also a 3-strand braid. Because of this, for these two representation not only $H_R^{8_{19}}$ are the same, but also the entire $\mathcal{H}_R^{8_{19}}$, as follows from the Table (for $R = [1]$): the coefficients h_R^Q are the same for both realizations.

We added two more lines to the Table, describing knot 10_{139} . The task is to explain the coincidence observed in [11]. 10_{139} can be considered as 5_2 with an attached torus braid:

$$\mathcal{B}^{10_{139}} = \mathcal{B}^{5_2} \left((\mathfrak{R} \otimes I)(I \otimes \mathfrak{R}) \right)^3 \quad (34)$$

and, thus, its HOMFLY polynomial (and even the superpolynomial) can be obtained from that of 5_2 by a "torus evolution" described in [11]. In our Table we demonstrate that the HOMFLY polynomial for (34) is indeed the same as for the conventional realization of 10_{139} .

The Table of HOMFLY polynomials including 3-strand knots with no more than 8 crossings plus knot 10₁₃₉ (see (34))

knot	(a_1, b_1, \dots)	S_3^*	S_{21}^*	S_{111}^*	A^2	A^0	A^{-2}
3 ₁	(-1,-1,-1,-1)	q^{-4}	-1	q^4	0	-1	$q^2 + q^{-2}$
4 ₁	(1,-1,1,-1)	1	$q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4}$	1	1	$-q^2 + 1 - q^{-2}$	1
5 ₂	(-1,1,-1,-3)	q^{-4}	$-(q^4 - q^2 + 1 - q^{-2} + q^{-4})$	q^4	-1	$q^2 - 1 + q^{-2}$	$q^2 - 1 + q^{-2}$
6 ₂	(1,-1,1,-3)	q^{-2}	$q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6}$	q^2	$q^2 - 1 + q^{-2}$	$-q^4 + q^2 - 2 + q^{-2} - q^{-4}$	$q^2 + q^{-2}$
6 ₃	(2,-1,1,-2)	1	$-(q^2 - 1 + q^{-2})(q^4 - q^2 + 1 - q^{-2} + q^{-4})$	1	$-q^2 + 1 - q^{-2}$	$q^4 - q^2 + 3 - q^{-2} + q^{-4}$	$-q^2 + 1 - q^{-2}$
7 ₃	(1,-1,1,5)	q^6	$-q^6 + q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4} - q^{-6}$	q^{-6}	$q^4 - q^2 + 1 - q^{-2} + q^{-4}$	$q^4 - q^2 + 2 + q^{-4} - q^{-2}$	$-q^2 - q^{-2}$
7 ₅	(-2,1,-1,-4)	q^6	$-(q^2 - 1 + q^{-2})(q^4 - q^2 + 1 - q^2 + 1)$	q^6	$-q^2 + 1 - q^{-2}$	$q^4 - 2q^2 + 2 - 2q^{-2} + q^{-4}$	$q^4 - q^2 + 2 + q^{-4} - q^{-2}$
8 ₂	(1,-1,1,-5)	q^{-4}	$q^8 - 2q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - 2q^{-6} + q^{-8}$	q^4	$q^4 - q^2 + 1 - q^{-2} + q^{-4}$	$-q^6 + q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4} - q^{-6}$	$q^4 + 1 + q^{-4}$
8 ₅	(-1,3,-1,3)	q^4	$q^8 - 2q^6 + 3q^4 - 4q^2 + 3 - 4q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}$	q^{-4}	$q^4 + 2 + q^{-4}$	$-q^6 + q^4 - 3q^2 + 1 - 3q^{-2} + q^{-4} - q^{-6}$	$q^4 - q^2 + 2 + q^{-4} - q^{-2}$
8 ₇	(-2,1,-1,4)	q^2	$-q^8 + 2q^6 - 3q^4 + 4q^2 - 5 + 4q^{-2} - 3q^{-4} + 2q^{-6} - q^{-8}$	q^{-2}	$-q^4 + q^2 - 1 + q^{-2} - q^{-4}$	$q^6 - q^4 + 3q^2 - 2 + 3q^{-2} - q^{-4} + q^{-6}$	$-q^4 + q^2 - 2 + q^{-2} - q^{-4}$
8 ₉	(3,-1,1,-3)	1	$(q^2 - 1 + q^{-2})(q^6 - q^4 + q^2 - 3 + q^{-2} - q^{-4} + q^{-6})$	1	$q^4 - q^2 + 2 + q^{-4} - q^{-2}$	$-q^6 + q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4} - q^{-6}$	$q^4 - q^2 + 2 + q^{-4} - q^{-2}$
8 ₁₀	(-2,2,-1,3)	q^2	$-q^8 + 2q^6 - 4q^4 + 5q^2 - 5 + 5q^{-2} - 4q^{-4} + 2q^{-6} - q^{-8}$	q^{-2}	$-q^4 + q^2 - 2 + q^{-2} - q^{-4}$	$q^6 - q^4 + 4q^2 - 2 + 4q^{-2} - q^{-4} + q^{-6}$	$-q^4 + q^2 - 3 + q^{-2} - q^{-4}$
8 ₁₆	(1,-1,1,-2,1,-2)	q^{-2}	$-q^8 + 3q^6 - 5q^4 + 6q^2 - 7 + 6q^{-2} - 5q^{-4} + 3q^{-6} - q^{-8}$	q^2	$-q^4 + 2q^2 - 3 + 2q^{-2} - q^{-4}$	$q^6 - 2q^4 + 4q^2 - 4 + 4q^{-2} - 2q^{-4} + q^{-6}$	$-q^4 + 2q^2 - 2 + 2q^{-2} - q^{-4}$
8 ₁₇	(2,-1,1,-1,1,-2)	1	$q^8 - 3q^6 + 5q^4 - 7q^2 + 7 - 7q^{-2} + 5q^{-4} - 3q^{-6} + q^{-8}$	1	$q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4}$	$-q^6 + 2q^4 - 4q^2 + 5 - 4q^{-2} + 2q^{-4} - q^{-6}$	$q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4}$
8 ₁₈	(1,-1,1,-1,1,-1,1,-1)	1	$q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}$	1	$q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4}$	$-q^6 + 3q^4 - 4q^2 + 7 - 4q^{-2} + 3q^{-4} - q^{-6}$	$q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4}$
8 ₁₉	(1,3,1,3)=[3,4] (1,1,1,1,1,1,1,1)	q^8 q^8	-1 -1	q^{-8} q^{-8}	$q^6 + q^2 + 1 + q^{-2} + q^{-6}$ $q^6 + q^2 + 1 + q^{-2} + q^{-6}$	$-q^4 - q^2 - 1 - q^{-2} - q^{-4}$ $-q^4 - q^2 - 1 - q^{-2} - q^{-4}$	1 1
8 ₂₀	(-1,-3,-1,3)	q^{-2}	$-q^6 + q^4 - q^2 + 1 - q^{-2} + q^{-4} - q^{-6}$	q^2	$-q^2 - q^{-2}$	$q^4 + 2 + q^{-4}$	$-q^2 + 1 - q^{-2}$
8 ₂₁	(-2,2,-1,-3)	q^{-4}	$q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6}$	q^4	$q^2 - 1 + q^{-2}$	$-q^4 + q^2 - 3 + q^{-2} - q^{-4}$	$2q^2 - 1 + 2q^{-2}$
10 ₁₃₉	(2,3,1,4) (1,-1,1,3,1,1,1,1,1,1)	q^{10} q^{10}	$-\frac{q^5 + q^{-5}}{q + q^{-1}}$ $-\frac{q^5 + q^{-5}}{q + q^{-1}}$	q^{-10} q^{-10}	$q^8 + q^4 + q^2 + q^{-2} + q^{-4} + q^{-8}$ $q^8 + q^4 + q^2 + q^{-2} + q^{-4} + q^{-8}$	$-q^6 - q^4 - 2 - q^{-4} - q^{-6}$ $-q^6 - q^4 - 2 - q^{-4} - q^{-6}$	$q^2 - 1 + q^{-2}$ $q^2 - 1 + q^{-2}$

Comments

1. Torus knots

Among arbitrary 3-strand braids described by arbitrary sequences a_1, b_1, \dots , one can select particular *series*.

The simplest example is the series of *torus* knots $[3, n]$ (for $n = 3k$ these are links), with $a_k = b_k = 1$, $k = 1, \dots, n$. In this case it makes sense to diagonalize not \mathfrak{R}_1 , but the product $\mathfrak{R}_1 \mathfrak{R}_2$, and the corresponding eigenvalues $r_Q^{torus}(i)$, $i = 1, \dots, \dim \mathcal{M}_Q = N_{12}^Q$ define the generic expression

$$H_{[1]}^{[3,n]} = \sum_Q \sum_{i=1}^{N_{12}^Q} (r_Q^{torus}(i))^n S_Q^* \quad (35)$$

Since

$$(\mathfrak{R} \otimes I)(I \otimes \mathfrak{R}) = \left(\begin{pmatrix} q^2 & & \\ & -\frac{1}{q[2]_q} & \frac{q\sqrt{[3]_q}}{[2]_q} \\ & -\frac{\sqrt{[3]_q}}{q[2]_q} & -\frac{q}{[2]_q} \\ & & & \left(-\frac{1}{q}\right)^2 \end{pmatrix} \right), \quad (36)$$

where the central 2×2 -block is just

$$\begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} \cdot \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \cdot \begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} \cdot \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} -\frac{1}{q[2]_q} & \frac{q\sqrt{[3]_q}}{[2]_q} \\ -\frac{\sqrt{[3]_q}}{q[2]_q} & -\frac{q}{[2]_q} \end{pmatrix} \quad (37)$$

one obtains

$$\begin{aligned} r_{[3]}^{torus} &= q^2, \\ r_{[21]}^{torus}(1) &= e^{2\pi i/3}, \quad r_{[21]}^{torus}(2) = e^{-2\pi i/3}, \\ r_{[111]}^{torus} &= -\frac{1}{q^2}, \end{aligned} \quad (38)$$

i.e.

$$\sum_{i=1}^{N_{12}^{[21]}} \left(r_{[21]}^{torus}(i) \right)^n = \left(e^{\frac{2\pi i}{3}} \right)^n + \left(e^{-\frac{2\pi i}{3}} \right)^n = 2 \cos \frac{2\pi n}{3} = \begin{cases} -1 & \text{for } n = 2k \pm 1, \\ 2 & \text{for } n = 2k \end{cases} \quad (39)$$

Thus, finally,

$$H_R^{[3,n]} = q^{2n} S_3^* + 2 \cos \frac{2\pi n}{3} \cdot S_{21}^* + q^{-2n} S_{111}^* \quad (40)$$

2. Other series

Of course, one can consider not only the series of torus knots, but any other series.

For example, take $a_k = 1$, $b_k = 3$, $k = 1, \dots, n$. In this case in the $[21]$ sector one has

$$(\mathfrak{R} \otimes I)(I \otimes \mathfrak{R})^3 = \begin{pmatrix} q^4 & & \\ & \frac{q^6 - q^4 - 1}{q^3[2]} & \frac{(q^4 - q^2 - 1)\sqrt{[3]_q}}{q[2]_q} \\ & -\frac{(q^4 - q^2 - 1)\sqrt{[3]_q}}{q^3[2]_q} & -\frac{q^6 - q^4 - 1}{q^3[2]_q} \\ & & & \frac{1}{q^4} \end{pmatrix} \quad (41)$$

with eigenvalues

$$e^{\pm 2\pi i/3} \quad (42)$$

just the same as for the torus case. However, in the sectors $[3]$ and $[111]$ we now have eigenvalues q^{4n} and q^{-4n} , instead of q^{2n} and q^{-2n} in the torus case, thus, expressions for the HOMFLY polynomials will be different.

Likewise, for the similar series $a_k = a$, $b_k = b$, $k = 1, \dots, n$ with other values of a and b one has $q^{(a+b)n}$ and $(-1/q)^{(a+b)n}$ in the $[3]$ and $[111]$ sectors respectively, while in the $[21]$ sector the situation is more complicated, the two eigenvalues being

$$r_{[21]}^{(a,b)}(\pm) = \frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 4(-1)^{a+b}}$$

$$\alpha = \frac{q^{a+b} + (-1)^{a+b}q^{-a-b} + [3]_q((-1)^b q^{a-b} + (-1)^a q^{b-a})}{[2]_q^2}$$

and only the first few eigenvalues are simple and do not contain square roots: $r_{[21]}^{(1,1)} = r_{[21]}^{(1,3)} = r_{[21]}^{(3,1)} = e^{\pm \frac{2\pi i}{3}}$, $r_{[21]}^{(1,2)} = r_{[21]}^{(2,1)} = \pm 1$ and $r_{[21]}^{(2,2)} = q^{\pm 2}$.

Of course, the square roots disappear from the sums $(r_{[21]}(+))^n + (r_{[21]}(-))^n$.

It deserves noting that the triviality of knots with (a, b) equal to $(0, 0)$ and $(1, 0)$ implies respectively the following identities between the Schur functions

$$\text{Tr}_{[21]} I = 2 \implies (S_3 + 2S_{21} + S_1^3) = S_1^3,$$

$$\text{Tr}_{[21]} \mathfrak{R} = q - \frac{1}{q} \implies \left(qS_3^* + \left(q - \frac{1}{q} \right) S_{21}^* - \frac{1}{q} S_{111}^* \right) = S_1^* \left(qS_2^* - \frac{1}{q} S_{11}^* \right), \quad (43)$$

...

3. Composite knots

If one takes arbitrary numbers of crossings, one can get not only prime knots which can not be represented as a sum of two or more independent knots (successively made one after another on the same string) or links from the tables. There is also another possibility: it can be composite of knots or links. For each combination of numbers of crossings one can evaluate the HOMFLY polynomial using our method (see Table below for examples). It is quite easy to distinguish these cases by the form of the HOMFLY polynomial. If it has several multipliers then it is a composite link or knot, and each of the multipliers is a HOMFLY polynomial of the corresponding knot or link (see (47) for examples). If it is a link then for each additional loop it has a multiplier $\frac{1}{(q-q^{-1})}$. Also links and knots can be oriented differently. For knots the change of the orientation is a quite simple procedure: one just should take the opposite to all the numbers of the crossings and it corresponds to the substitution of A^{-1} instead of A into the HOMFLY polynomial. With the links, however, the situation is more complicated, because one can change the direction of some of the loops. In this case, the HOMFLY

polynomial changes non-trivially (see (49) for example: these two links have the same picture, but the different orientation), but the change of numbers of crossings is still quite trivial: one should change all the numbers which the changed loop involves.

The HOMFLY polynomials of the composite knots are always factorized:

$$H_R^{\mathcal{K}_1 \# \mathcal{K}_2} = \frac{H_R^{\mathcal{K}_1} H_R^{\mathcal{K}_2}}{H_R^\emptyset} \quad (44)$$

This is slightly different from the decomposition of the HOMFLY polynomials for the disjoint unification of two knots,

$$H_R^{\mathcal{K}_1 \cup \mathcal{K}_2} = H_R^{\mathcal{K}_1} H_R^{\mathcal{K}_2} \quad (45)$$

The simplest examples of composite knots are made out of the 3-strand braids. In particular, composite are obviously the knots with two non-vanishing odd parameters a_1 and b_1 , they are in fact a composition of two 2-strand knots with parameters a_1 and b_1 respectively. Similarly, $\{a_1, b_1, c_1\}$ and $\{a_1, b_1, c_1, d_1\}$ will be obviously composite knots (or links) for 4 and 5 strands respectively.

Of course, the set of composite 3-strand knots is not exhausted by the set (a_1, b_1) . In particular, the "torus descendant" of 4_1 , which was considered in [11], in our present notation it is a 3-strand braid $(1, -1, 1, -1, 1, 1, 1, 1, 1, 1)$, is equivalent to $(3, 3)$:

$$H_{[1]}^{(1, -1, 1, -1, 1, 1, 1, 1, 1, 1)} = H_{[1]}^{(3, 3)} = \left(A(q^2 + q^{-2}) - A^{-1} \right)^2 S_1^* = \frac{\left(H_{[1]}^{3_1} \right)^2}{S_1^*} \quad (46)$$

These are examples of the simplest composite knots (the sequences a_1, b_1, \dots are written here in braces, $\#$ denotes composition of knots and we use dual notations for the torus knots):

$$\text{Knot} = \{3, 3\} = T[2, 3] \# T[2, 3] : \quad H = \left(A(q^2 + q^{-2}) - A^{-1} \right)^2 \quad (47)$$

$$\text{Knot} = \{3, 4\} = T[2, 3] \# T[2, 4] :$$

$$H = \frac{1}{q - q^{-1}} \left(A(q^2 + q^{-2}) - A^{-1} \right) \left(A(q^4 - q^2 + 1 - q^{-2} + q^{-4}) - A^{-1}(q^2 - 1 + q^{-2}) \right) \quad (48)$$

$$\text{Knot} = \{2, 1, 1, 1\} = T[2, 4] : \quad H = \frac{1}{q - q^{-1}} \left(A^2(q^4 - q^2 + 1 - q^{-2} + q^{-4}) - (q^2 - 1 + q^{-2}) \right)$$

$$\text{Knot} = \{2, 1, -1, 1\} = L4a1 : \quad H = \frac{1}{q - q^{-1}} \left(A^2(q^2 - 2 + q^{-2}) + (q^2 - 1 + q^{-2}) - A^{-2} \right) \quad (49)$$

and

$$H^{T[2, 3]} = A(q^2 + q^{-2}) - A^{-1} \quad (50)$$

$$H^{T[2, 4]} = \frac{1}{q - q^{-1}} \left(A(q^4 - q^2 + 1 - q^{-2} + q^{-4}) - A^{-1}(q^2 - 1 + q^{-2}) \right)$$

More examples of knots/links with a_1, b_1, a_2, b_2 not exceeding 2 in absolute value can be found in the following Table:

knots			
0_1	1,-1,-1,-1	1,1,-1,-1	1,1,-2,2
3_1	-1,-1,-1,-1	1,1,-2,-2	
4_1	1,-1,1,-1	1,-1,-2,2	
5_1	1,1,2,2		
5_2	1,-1,-2,-2		
6_3	1,-2,2,-1		
links			
$2 \times 0_1$	1,1,1,-2		
$L2a1$	1,-1,-1,-2	1,1,-1,-2	
$L4a1$	1,-1,-2,-1	2,-2,-1,-2	
$T(2,4)$	2,1,1,1		
$L5a1$	1,-1,1,-2	1,2,-2,-2	
$L6n1$	1,2,1,-2		
$L7n1$	-1,-2,-2,-2		
$L7n2$	2,-1,-2,-2		
$4L2a1\#L2a1$	1,2,-1,2		
$L2a1\#L2a1'$	1,2,-1,-2		

A large variety of links and composite knots explains why the sequences a_1, b_1, \dots appearing in our Table of 3-strand knots above are not arbitrary. The list actually refers to the prime knots.

As follows from the observation in [11] on the "descendant" $(1, -1|1, -1|1, 1|1, 1|1, 1)$ of the knot $4_3 = (1, -1|1, -1)$, which (descendant) appears to be a composite knot, the "*superpolynomials*" for composite knots are more complicated than those for the prime ones. This is not very surprising because of the special role of the unknot H_R^\emptyset in the decomposition rule (44), and unknot superpolynomial is a subtle issue already by itself.

4 Systematic description of \mathcal{U} -matrices

Before proceeding to the 4-strand and 5-strand knots, we now provide a general construction of the mixing \mathcal{U} -matrices. In fact, as we already noted they are nothing but the Racah coefficients, however, as we shall see, there might be recovered a deep hierarchical structure expressing the relevant compositions of the Racah coefficients through simpler ingredients, which seem to possess a general (universal) description.

In application to braid calculus it will be convenient to build \mathcal{U} -matrices hierarchically, and the fusion pattern of representations can be presented by rooted tree diagrams. The key point is that the elementary \mathfrak{R} -matrices appear only at the first level of hierarchy, when just *two* representations R merge, while mixing arises at different levels of the tree. We count levels from the top, not from the root. All our mixings are actually described by 2×2 matrices. Moreover, they are universal in the following sense:

- everything unmixed at one level remains unmixed at the next levels
- all descendants of a given mixing are the same at all next levels

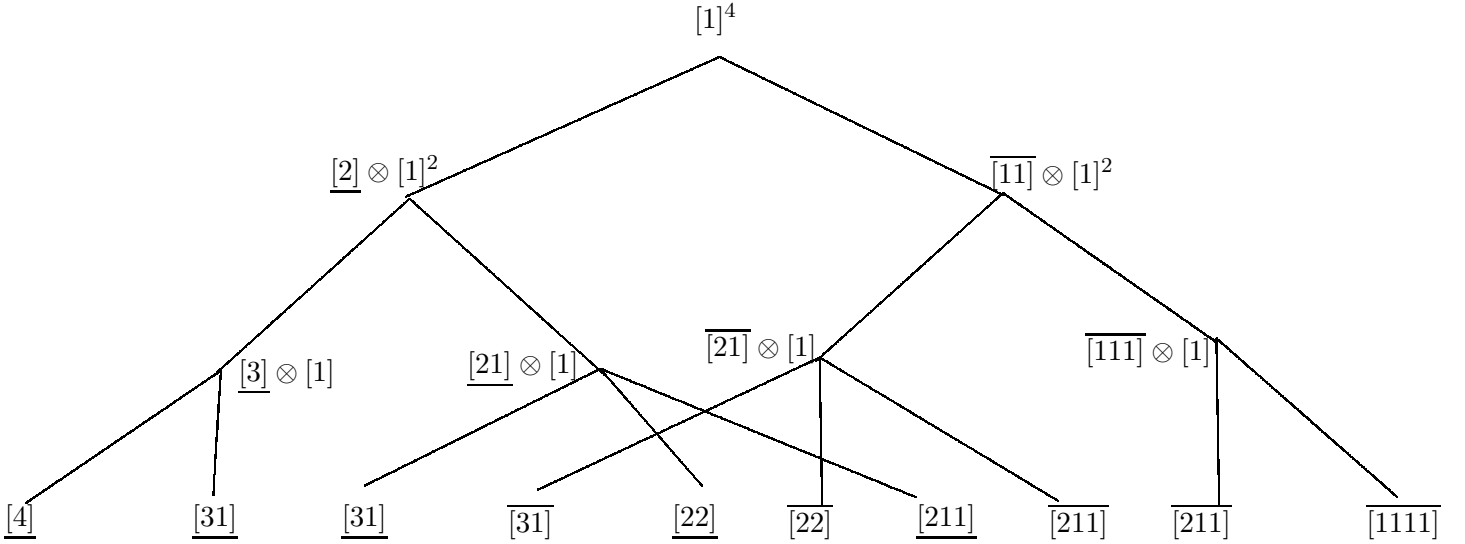
Then the full \mathcal{U} -matrices are represented as the ordered products of elementary ones, arising at different levels. Our simple examples (for $m \leq 5$) involve just three levels, and we denote the

corresponding elementary factors by U , V and W . For $R = [1]$ all the three will be essentially 2×2 matrices. Of course, one can ignore this additional hierarchical structure and evaluate the full \mathcal{U} matrices directly.

It is sufficient to deal with $SU_q(2)$ representations in order to study all the representations given by the Young diagrams with no more than two rows. After that one can use the symmetry $R, q \rightarrow R', -1/q$ in order to find expressions for the two-column representations. For $[1]^{\otimes m}$ the $SU_q(2)$ is insufficient starting from $m = 5$, when $[311]$ arises.

We study the mixing, looking explicitly at the highest weights of the corresponding representations. The highest weights, in turn, show up at different *grades* of the Verma module constructions.

Let us look at the example of tensor product of four fundamental representations $[1]^{\otimes 4}$. It can be expanded, for instance, in the following way:



At the lowest level of this picture, there emerge three representations $[31]$ and $[211]$, two representations $[22]$, while representations $[4]$ and $[1111]$ emerge only once. We throw away a half of the representations that can be obtained by transposing the corresponding Young diagrams.

Now one could construct a similar expansion, fusing at the first step, e.g., the two rightmost fundamental representations etc. That would lead to the same set of representations, however, the three representations $[31]$ emerging would be now linear combinations of those obtained with the first pattern of fusion above. The (orthogonal) matrix that relates representations $[31]$ in two different patterns of fusion is constructed from the set of the Racah coefficients ($6j$ -symbols) and is exactly the U -matrix we are looking for. (Similarly, we are interested in the matrix rotating the two representations $[22]$.)

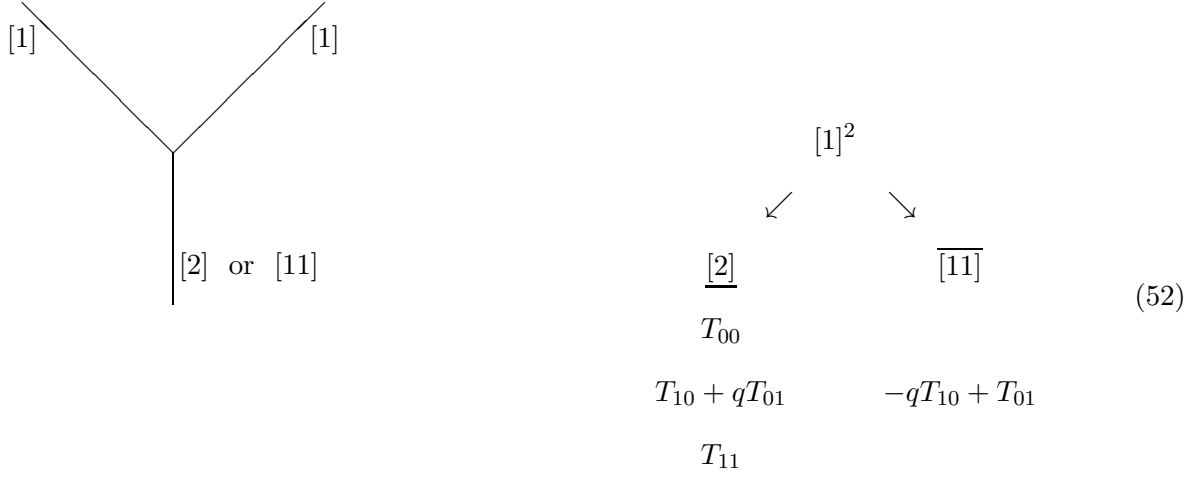
In what follows we find such matrices, constructing manifestly the vectors in the representations and using the action of lowering and raising operators at the tensor product:

$$\downarrow \quad \Delta(F) = q^{2H} \otimes F + F \otimes I, \quad \uparrow \quad \Delta(E) = I \otimes E + E \otimes q^{-2H} \quad (51)$$

We begin with the highest weight $T_{++++} = T_{0000}$ and look at $1^{\otimes m}$ at $m = 2, 3, 4, 5$. At level one we get the pattern of the highest weights, relevant for description of the mixing of $[m-1, 1]$ representations. Likewise, at level two we get the description of mixing in the $[m-2, 2]$ sector.

4.1 $m = 2$

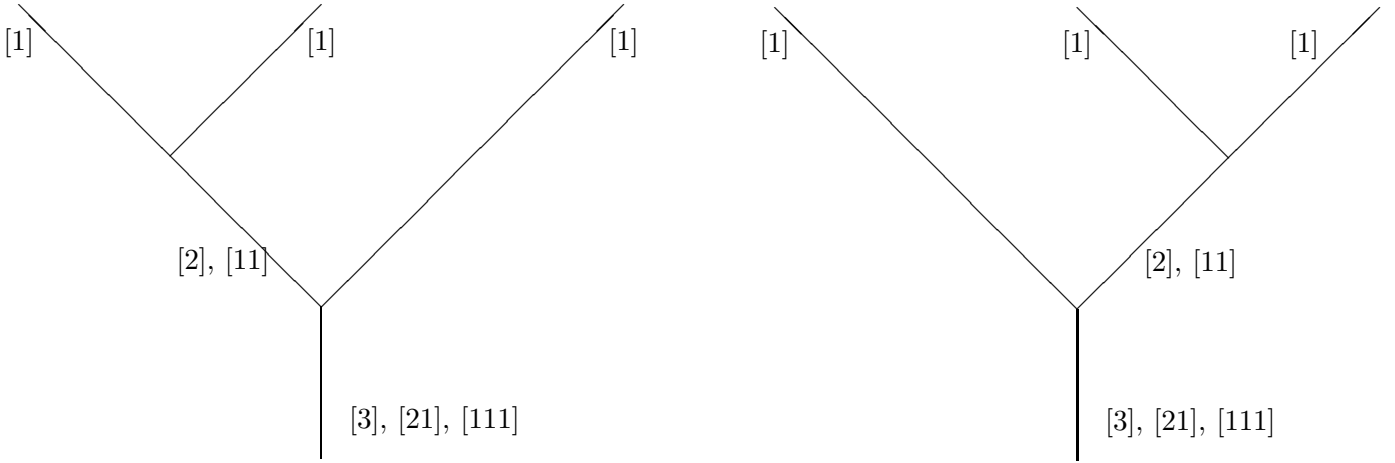
In this case, there is only one pattern of fusion, the vectors in the representations being manifestly indicated at the picture:



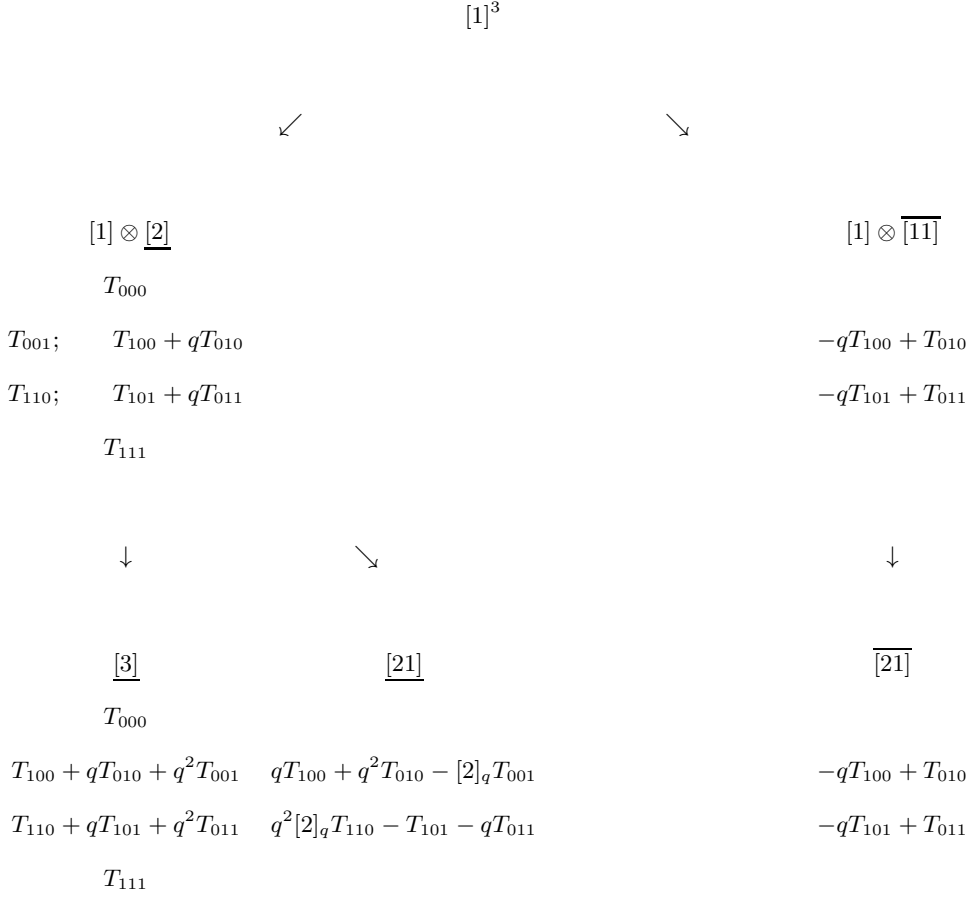
The diagram shows a fusion tree for $m=2$. It starts with two $[1]$ representations at the top, which fuse into a single representation $[2] \text{ or } [11]$ at the bottom. To the right, a more detailed fusion tree is shown, labeled (52). It starts with $[1]^2$ at the top, which branches into two paths. The left path goes through $\underline{[2]}$ (labeled T_{00}) to $T_{10} + qT_{01}$ (labeled T_{11}). The right path goes through $\overline{[11]}$ to $-qT_{10} + T_{01}$.

4.2 $m = 3$

In this case, there are two possible fusion trees:



Let us list the vectors in the representations emerging in the right tree through fusions in the $SU(2)$ case (so that $[111]$ does not emerge):



We did not normalize the vectors here.

One can see at this scheme how the vectors at each grade are parted between representations. For instance, at the first level of the hierarchy one just has one grade zero vector T_{000} , three grade one vectors T_{001} , T_{010} and T_{100} , three grade two vectors T_{011} , T_{110} and T_{101} and one grade three vector T_{111} . These vectors are further distributed between various representations. At the second level of the hierarchy, T_{000} goes to $[1] \otimes [2]$, while the three grade one vectors, T_{001} , T_{010} and T_{100} are distributed between $[1] \otimes [2]$ and $[1] \otimes [11]$ etc.

From now on, we are interested in the vectors in irreducible representations only, i.e. at the last level of the hierarchy. In particular, the content of irreps for the left tree is

$$\begin{array}{ll}
& T_{000} \\
[1]^3 \rightarrow \underline{[2]} \otimes [1] \rightarrow \underline{[3]} : & \begin{array}{l} T_{100} + qT_{010} + q^2T_{001} \\ T_{110} + qT_{101} + q^2T_{011} \end{array} \\
& T_{111} \\
[1]^3 \rightarrow \underline{[2]} \otimes [1] \rightarrow \underline{[21]} : & \begin{array}{l} -q^2[2]_q T_{100} + T_{010} + qT_{001} \\ -[2]_q T_{011} + q^2T_{101} + qT_{110} \end{array} \\
[1]^3 \rightarrow \overline{[11]} \otimes [1] \rightarrow \overline{[21]} : & \begin{array}{l} -qT_{010} + T_{001} \\ -qT_{110} + T_{101} \end{array}
\end{array}$$

Looking at the elements of the representations at the first grade, one immediately read out expressions for the highest weights of the two representations $[21]$ at the left and right trees accordingly:

$$\begin{aligned} \text{left tree} \longrightarrow & \left\{ \begin{array}{ll} \underline{[21]} : & -q^2[2]_q T_{100} + T_{010} + qT_{001}; & ||\dots||^2 = q^3[2]_q[3]_q \\ \overline{[21]} : & -qT_{010} + T_{001}; & ||\dots||^2 = q[2]_q \end{array} \right. \\ \text{right tree} \longrightarrow & \left\{ \begin{array}{ll} \underline{[21]} : & qT_{100} + q^2T_{010} - [2]_q T_{001}; & ||\dots||^2 = q[2]_q[3]_q \\ \overline{[21]} : & -qT_{100} + T_{010}; & ||\dots||^2 = q[2]_q \end{array} \right. \end{aligned}$$

The last column contains the squares of norms of these highest weights (the sums of squares of the coefficients³). If highest weights are divided by these norms, the mixing matrices are orthogonal. This is the way we define them in what follows. In particular, in this way one immediately obtains the orthogonal 2×2 mixing matrix U :

$$U = \left(\left(\begin{array}{cc} \frac{1}{[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{1}{[2]_q} \end{array} \right) \right) \quad (53)$$

which is exactly (29).

4.3 $m = 4$

In this case, there are four possible fusion trees.

³ Throughout the paper we assume that the norms are defined in the naive way: :

$$\langle T_{k_1 k_2 \dots} | T_{k'_1 k'_2 \dots} \rangle = \prod_{i=1}^m \delta_{k_i k'_i}$$

In fact, the scalar product is a little more complicated, if generators E and F with comultiplication (51) are conjugate (the representation is unitary). Indeed,

$$\langle \Delta(E)(A \otimes B) | C \otimes D \rangle = \langle A \otimes B | \Delta(F)(C \otimes D) \rangle$$

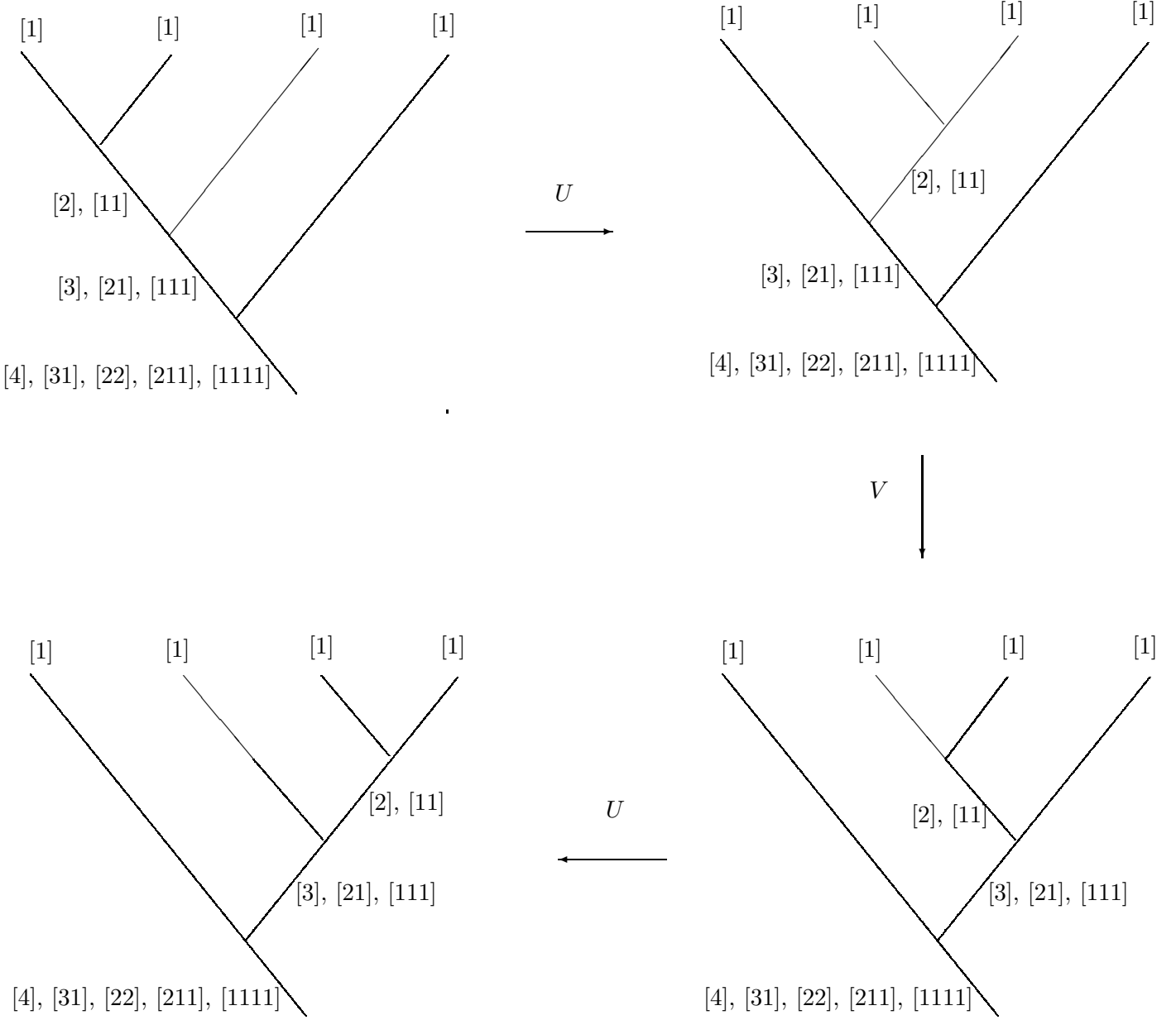
implies that

$$\langle A \otimes B | C \otimes D \rangle = q^{2h_A h_D} \langle A | C \rangle \langle B | D \rangle$$

and in general

$$\langle \otimes_{i=1}^m A_i | \otimes_{i=1}^m C_i \rangle = q^{2\sum_{i < j} h_{A_i} h_{C_j}} \prod_{i=1}^m \langle A_i | C_i \rangle$$

However, for $R = [1]$ the "cocycle" factor, though non-trivial, can be neglected in all our calculations, because in this case of $p = 1$ (only!) the norms of all the elements of the same grade in Verma module are the same: $||T_{k_1 k_2 \dots}||$ depends only on the sum $\sum_i k_i$ and not on a particular choice of $\{k_i\}$.



Since we deal here with $SU(2)$ only, representations $[111]$, $[211]$ and $[1111]$ are not seen, however, their transposed are. Hence, the complete results in the 4-strand case can be obtained from the $SU(2)$ group only (and the fundamental representations). The vectors in the representations in the first tree are

$$\begin{aligned}
 & T_{0000} \\
 & T_{1000} + qT_{0100} + q^2T_{0010} + q^3T_{0001} \\
 [1]^4 \rightarrow \underline{[2]} \otimes [1]^2 \rightarrow \underline{[3]} \times [1] \rightarrow \underline{[4]} : & T_{1100} + qT_{1010} + q^2T_{0110} + q^2T_{1001} + q^3T_{0101} + q^4T_{0011} \\
 & T_{1110} + qT_{1101} + q^2T_{1011} + q^3T_{0111} \\
 & T_{1111}
 \end{aligned}$$

$$\begin{aligned}
& T_{1000} + qT_{0100} + q^2T_{0010} - \frac{[3]_q}{q}T_{0001} \\
[1]^4 \rightarrow \underline{[2]} \otimes [1]^2 \rightarrow \underline{[3]} \times [1] \rightarrow \underline{[31]} : & \quad T_{1100} + qT_{1010} + q^2T_{0110} - \frac{1}{q^2}T_{1001} - \frac{1}{q}T_{0101} - T_{0011} \\
& \quad T_{1110} - \frac{1}{q^3}T_{1101} - \frac{1}{q^2}T_{1011} - \frac{1}{q}T_{0111} \\
& T_{1000} + qT_{0100} - \frac{[2]_q}{q}T_{0010} \\
[1]^4 \rightarrow \underline{[2]} \otimes [1]^2 \rightarrow \underline{[21]} \times [1] \rightarrow \underline{[31]} : & \quad [2]_q T_{1100} - \frac{1}{q^2}T_{1010} - \frac{1}{q}T_{0110} + qT_{1001} + q^2T_{0101} - [2]_q T_{0011} \\
& \quad [2]_q T_{1101} - \frac{1}{q^2}T_{1011} - \frac{1}{q}T_{0111} \\
& -qT_{1000} + T_{0100} \\
[1]^4 \rightarrow \overline{[11]} \otimes [1]^2 \rightarrow \overline{[21]} \times [1] \rightarrow \overline{[31]} : & \quad -qT_{1010} + T_{0110} - q^2T_{1001} + qT_{0101} \\
& \quad -qT_{1011} + T_{0111}
\end{aligned}$$

Looking at the elements of the representations at the first grade, one immediately reads off expressions for the three highest weights of the three representations [31]:

$$\begin{array}{lll}
\underline{[31]} & T_{1000} + qT_{0100} + q^2T_{0010} - \frac{[3]_q}{q}T_{0001} & q^{-1}[2]_q[3]_q(q^2 + q^{-2}) \\
\underline{[31]} & T_{1000} + qT_{0100} - \frac{[2]_q}{q}T_{0010} & q^{-1}[2]_q[3]_q \\
\overline{[31]} & -qT_{1000} + T_{0100} & q[2]_q
\end{array}$$

At the second grade the four representations $[4] + 3[31]$ do not exhaust all of the tensor T , even for $SU(2)$. There are actually six elements of T at this grade, and only four belong to those four representations. Thus, there are two more highest weights: those of representations $[22]$ (as to $[211]$ and $[1111]$, they are "non-observable" while we deal with $SU(2)$). These two new weights are:

$$\begin{aligned}
\underline{[21]} \otimes [1] & \longrightarrow \underline{[22]} & q[2]_q T_{1100} - \frac{1}{q}T_{1010} - T_{0110} - T_{1001} - qT_{0101} + \frac{[2]_q}{q}T_{0011} & [2]_q^2[3]_q \\
\overline{[21]} \otimes [1] & \longrightarrow \overline{[22]} & -qT_{1010} + T_{0110} + T_{1001} - \frac{1}{q}T_{0101} & [2]_q^2
\end{aligned}$$

Now we switch to the other tree related with the just considered one by the mixing matrix U (see the trees in the diagram above) and obtain the following highest vectors in the representations [31]:

$$\begin{array}{lll}
\underline{[31]} & T_{1000} + qT_{0100} + q^2T_{0010} - \frac{[3]_q}{q}T_{0001} & q^{-1}[2]_q[3]_q(q^2 + q^{-2}) \\
\underline{[31]} & -q^2[2]_q T_{1000} + T_{0100} + qT_{0010} & q^3[2]_q[3]_q \\
\overline{[31]} & -qT_{0100} + T_{0010} & q[2]_q
\end{array}$$

Applying further the matrix V , we come to the next tree with the highest vectors in the representations $[31]$ as follows:

$$\begin{array}{ll} \underline{[31]} & -q^2[3]_q T_{1000} + T_{0100} + qT_{0010} + q^2 T_{0001} \quad q^5[2]_q[3]_q(q^2 + q^{-2}) \\ [31] & T_{0100} + qT_{0010} - \frac{[2]_q}{q} T_{0001} \quad q^{-1}[2]_q[3]_q \\ \overline{[31]} & -qT_{0100} + T_{0010} \quad q[2]_q \end{array}$$

The last tree can be obtained with action of the same matrix U , the highest vectors being

$$\begin{array}{ll} \underline{[31]} & -q^3[3]_q T_{1000} + T_{0100} + qT_{0010} + q^2 T_{0001} \quad q^5[2]_q[3]_q(q^2 + q^{-2}) \\ [31] & -q^2[2]_q T_{0100} + T_{0010} + qT_{0001} \quad q^{-1}[2]_q[3]_q \\ \overline{[31]} & -qT_{0010} + T_{0001} \quad q[2]_q \end{array}$$

Similarly to the previous subsection, from these manifest expressions for the vectors one can read off the mixing matrices of rotation:

$$U = \left(\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ 0 & -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{1}{[2]_q} \end{array} \right) \right) \quad (54)$$

$$V = \left(\left(\begin{array}{ccc} \frac{1}{[3]_q} & \frac{[2]_q \sqrt{q^2 + q^{-2}}}{[3]_q} & 0 \\ -\frac{[2]_q \sqrt{q^2 + q^{-2}}}{[3]_q} & \frac{1}{[3]_q} & 0 \\ 0 & 0 & 1 \end{array} \right) \right) \quad (55)$$

so that

$$\begin{aligned} \mathfrak{R}_1^{[31]} &= \left(\left(\begin{array}{ccc} q & & \\ & q & \\ & & -\frac{1}{q} \end{array} \right) \right), \\ \mathfrak{R}_2^{[31]} = U \mathfrak{R}_1^{[31]} \tilde{U} &= \left(\left(\begin{array}{ccc} q & 0 & 0 \\ 0 & -\frac{1}{q^2[2]_q} & -\frac{\sqrt{[3]_q}}{[2]_q} \\ 0 & -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} \end{array} \right) \right), \\ \mathfrak{R}_3^{[31]} = UV \mathfrak{R}_2^{[31]} \widetilde{UV} &= \left(\left(\begin{array}{ccc} -\frac{1}{q^3[3]_q} & -\frac{[2]_q \sqrt{q^2 + q^{-2}}}{[3]_q} & 0 \\ -\frac{[2]_q \sqrt{q^2 + q^{-2}}}{[3]_q} & \frac{q^3}{[3]_q} & 0 \\ 0 & 0 & q \end{array} \right) \right) \end{aligned} \quad (56)$$

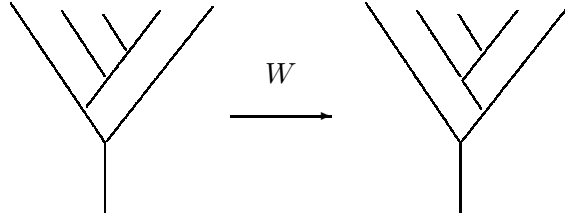
and

$$\begin{aligned}
\mathfrak{R}_1^{[22]} &= \left(\begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} \right), \\
\mathfrak{R}_2^{[22]} = U^{[22]} \mathfrak{R}_1^{[22]} \tilde{U}^{[22]} &= \left(\begin{pmatrix} -\frac{1}{q^2[2]_q} & -\frac{\sqrt{[3]_q}}{[2]_q} \\ -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} \end{pmatrix} \right), \\
\mathfrak{R}_3^{[22]} = \tilde{U}^{[22]} \mathfrak{R}_2^{[22]} U^{[22]} &= \left(\begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} \right), \text{ since } V^{[22]} = \left(U^{[22]} \right)^{-2} = \left(\begin{pmatrix} -\frac{[4]}{[2]^3} & -2\frac{\sqrt{[3]_q}}{[2]^2} \\ 2\frac{\sqrt{[3]_q}}{[2]^2} & -\frac{[4]}{[2]^3} \end{pmatrix} \right)
\end{aligned} \tag{57}$$

where tilde implies the transposed matrix (or inverse, since all they are orthogonal).

4.4 $m = 5$

For $m = 5$ there is one transition between the trees of a new type, W :



The highest vectors in the four representations [41] in these two trees correspondingly look like

$$\begin{array}{ll}
\text{[41]} & T_{10000} + qT_{01000} + q^2T_{00100} + q^3T_{00010} - \frac{[4]_q}{q}T_{00001} \quad q^{-1}[2]_q[5]_q(q^2 + q^{-2}) \\
\text{[41]} & -q^3[3]_qT_{10000} + T_{01000} + qT_{00100} + q^2T_{00010} \quad q^5[2]_q[3]_q(q^2 + q^{-2}) \\
\text{[41]} & -q^2[2]_qT_{01000} + T_{00100} + qT_{00010} \quad q^3[2]_q[3]_q \\
\overline{\text{[41]}} & -qT_{00100} + T_{00010} \quad q[2]_q
\end{array}$$

and

$$\begin{array}{ll}
\text{[41]} & -q^4[4]_qT_{10000} + T_{01000} + qT_{00100} + q^2T_{00010} + q^3T_{00001} \quad q^7[2]_q[5]_q(q^2 + q^{-2}) \\
\text{[41]} & T_{01000} + qT_{00100} + q^2T_{00010} - \frac{[3]_q}{q}T_{00001} \quad q^{-1}[2]_q[3]_q(q^2 + q^{-2}) \\
\text{[41]} & -q^2[2]_qT_{01000} + T_{00100} + qT_{00010} \quad q^3[2]_q[3]_q \\
\overline{\text{[41]}} & -qT_{00100} + T_{00010} \quad q[2]_q
\end{array}$$

They are related by the new 4×4 matrix W :

$$W = \begin{pmatrix} \begin{pmatrix} \frac{1}{[4]_q} & \frac{\sqrt{[3]_q[5]_q}}{[4]_q} & 0 & 0 \\ -\frac{\sqrt{[3]_q[5]_q}}{[4]_q} & \frac{1}{[4]_q} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \quad (58)$$

In general in the sector $[m-1, 1] \in [1]^{\otimes m}$ one has $(m-1) \times (m-1)$ matrices

$$U_{(k-1)} = \begin{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & & & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{[k]_q} & \frac{\sqrt{[k]_q^2 - 1}}{[k]_q} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{[k]_q^2 - 1}}{[k]_q} & \frac{1}{[k]_q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \dots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{pmatrix} \quad (59)$$

where non-unit 2×2 matrix stands in the rows and columns $m-k, m-k+1$ of the full matrix, $k = 2, 3, \dots, m-1$.

In the previous examples $U = U_{(1)}$, $V = U_{(2)}$, $W = U_{(3)}$:

$$U = \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \dots & & & & \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ 0 & 0 & 0 & -\frac{\sqrt{[3]_q}}{[2]_q} & \frac{1}{[2]_q} \end{pmatrix} \right), \quad (60)$$

$$V = \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \dots & & & & \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{[3]_q} & \frac{[2]_q \sqrt{q^2 + q^{-2}}}{[3]_q} & 0 \\ 0 & 0 & -\frac{[2]_q \sqrt{q^2 + q^{-2}}}{[3]_q} & \frac{1}{[3]_q} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

$$W = \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \frac{1}{[4]_q} & \frac{\sqrt{[3]_q [5]_q}}{[4]_q} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{[3]_q [5]_q}}{[4]_q} & \frac{1}{[4]_q} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right) \quad (61)$$

since

$$\begin{aligned} [2]_q^2 - 1 &= [3]_q, \\ [3]_q^2 - 1 &= [2]_q^2 [2]_{q^2} = [2]_q [4]_q, \\ [4]_q^2 - 1 &= [3]_q [5]_q, \\ [5]_q^2 - 1 &= [2]_q [3]_q [2]_{q^2} [2]_{q^3} = [4]_q [6]_q, \\ &\dots, \\ [k]_q^2 - 1 &= [k+1]_q [k-1]_q \end{aligned} \quad (62)$$

5 4-strand braids

In order to calculate $H_{[1]}$ in the case of four strands, one needs to deal with traces

$$\text{tr}_R \hat{\mathfrak{R}}_1^{a_1} \hat{\mathfrak{R}}_2^{b_1} \hat{\mathfrak{R}}_3^{c_1} \hat{\mathfrak{R}}_1^{a_2} \hat{\mathfrak{R}}_2^{b_2} \hat{\mathfrak{R}}_3^{c_2} \dots \quad (63)$$

over five possible representations R emerging in the decomposition

$$[1]^4 = [4] + 3[31] + 2[22] + 3[211] + [1111] \quad (64)$$

i.e. with the sum

$$\begin{aligned} H_{[1]} &= \text{Tr} \left\{ \left(\mathfrak{R} \otimes I \otimes I \right)^{a_1} \left(I \otimes \mathfrak{R} \otimes I \right)^{b_1} \left(I \otimes I \otimes \mathfrak{R} \right)^{c_1} \left(\mathfrak{R} \otimes I \otimes I \right)^{a_2} \left(I \otimes \mathfrak{R} \otimes I \right)^{b_2} \left(I \otimes I \otimes \mathfrak{R} \right)^{c_2} \dots \right\} = \\ &= \sum_{Q=[1111],[211], [22],[31],[4]} \text{tr} \left\{ \left(\hat{\mathfrak{R}}_1^Q \right)^{a_1} \left(\hat{\mathfrak{R}}_2^Q \right)^{b_1} \left(\hat{\mathfrak{R}}_3^Q \right)^{c_1} \left(\hat{\mathfrak{R}}_1^Q \right)^{a_2} \left(\hat{\mathfrak{R}}_2^Q \right)^{b_2} \left(\hat{\mathfrak{R}}_3^Q \right)^{c_2} \dots \right\} = \\ &= \sum_{Q=[1111],[211], [22],[31],[4]} \text{tr} \left\{ \left(\hat{\mathfrak{R}}_1^Q \right)^{a_1} \hat{U}^Q \left(\hat{\mathfrak{R}}_1^Q \right)^{b_1} \hat{V}^Q \hat{U}^Q \left(\hat{\mathfrak{R}}_1^Q \right)^{c_1} \hat{U}^Q \widetilde{\hat{V}^Q} \hat{U}^Q \times \right. \\ &\quad \left. \times \left(\hat{\mathfrak{R}}_1^Q \right)^{a_2} \hat{U}^Q \left(\hat{\mathfrak{R}}_1^Q \right)^{b_2} \hat{V}^Q \hat{U}^Q \left(\hat{\mathfrak{R}}_1^Q \right)^{c_2} \hat{U}^Q \widetilde{\hat{V}^Q} \hat{U}^Q \dots \right\} \end{aligned} \quad (65)$$

since

$$\hat{\mathfrak{R}}_2^Q = \hat{U}^Q \hat{\mathfrak{R}}_1^Q \widetilde{\hat{U}^Q}, \quad \hat{\mathfrak{R}}_3^Q = \hat{U}^Q \hat{V}^Q \hat{U}^Q \hat{\mathfrak{R}}_1^Q \widetilde{\hat{U}^Q \hat{V}^Q \hat{U}^Q} \quad (66)$$

The corresponding manifest expressions for \mathfrak{R} - and U -, V -matrices can be found in the previous section.

In particular, the contributions from $[4]$ and $[1111]$ sectors to the HOMFLY polynomials are trivial: there are no mixing matrices and all what we get is simply

$$q^{a_1+b_1+c_1+a_2+b_2+c_2+\dots} S_4^* + (-1/q)^{a_1+b_1+c_1+a_2+b_2+c_2+\dots} S_{1111}^* \quad (67)$$

Evaluation of contribution from $[22]$ is straightforward: the 2×2 \mathfrak{R} -matrices are actually given by the same mixing matrix as that in the $[21]$ sector for the 3-strand braids, hence, by the same \mathfrak{R} -matrix at the adjacent strand, (57b) and trivial \mathfrak{R} -matrix at the non-adjacent strand (57c).

However, the contribution to, say, torus knots is different, because now the \mathfrak{R} -matrix enters three times instead of two in the $[21]$ case, and

$$\text{tr}_{[22]} \left(\hat{\mathfrak{R}}_1 \hat{\mathfrak{R}}_2 \hat{\mathfrak{R}}_3 \right)^n = 1^n + (-1)^n = \begin{cases} 0 & \text{for odd } n \neq 0 \bmod 2 \\ 2 & \text{for even } n = 0 \bmod 2 \end{cases} \quad (68)$$

Similarly, the $[31]$ -contribution is done with using the \mathfrak{R} -matrices from (56). For torus knots

$$\text{tr}_{[31]} \left(\hat{\mathfrak{R}}_1 \hat{\mathfrak{R}}_2 \hat{\mathfrak{R}}_3 \right)^n = \sum_{k=1..L-1} e^{\frac{2\pi i k n}{L}} q^n \Big|_{L=4} = \begin{cases} -q^n & \text{for } n \neq 0 \bmod 4 \\ 3q^n & \text{for } n = 0 \bmod 4 \end{cases} \quad (69)$$

The contribution of the $[211]$ sector can be obtained from that of $[31]$ by a simple substitution $q \rightarrow (-1/q)$ (in particular, $S_{211}^*(q) = S_{31}^*(-1/q)$).

The Table of HOMFLY polynomials including 4-strand knots with 6, 7, 8 crossings

knot	(a_1, b_1, \dots)	S_4^*	S_{31}^*	S_{22}^*	S_{211}^*	S_{1111}^*
6 ₁	(1,-1,1,1,-1,-2)	q^{-1}	$q^5 - q^3 - q^1 + q^{-1} - 2q^{-3} + q^{-5}$	$q^{-1} - q^1$	$-q^5 + 2q^3 - q^1 + q^{-1} + q^{-3} - q^{-5}$	$(-q)$
7 ₂	(-1,1,0,-1,-1,1,0,-1,-3)	q^{-5}	$-q^3 + 3q^1 - 4q^{-1} + 3q^{-3} - 2q^{-5}$	$-q^7 + q^5 - 2q^3 + 3q^1 - 3q^{-1} + 2q^{-3} - q^{-5} + q^{-7}$	$2q^5 - 3q^3 + 4q^1 - 3q^{-1} + q^{-3}$	$-q^5$
7 ₄	(1,-1,0,1,2,-1,0,1,2)	q^5	$-3q^5 + 5q^3 - 6q^1 + 5q^{-1} - 2q^{-3}$	$(q - q^{-1})(q^6 - q^4 + 3q^2 - 1 + 3q^{-2} - q^{-4} + q^{-6})$	$2q^3 - 5q^1 + 6q^{-1} - 5q^{-3} + 3q^{-5}$	$-q^{-5}$
7 ₆	(-1,1,-1,-1,1,-2)	q^{-3}	$-q^3 + 3q^1 - 5q^{-1} + 5q^{-3} - 4q^{-5} + q^{-7}$	$-q^7 + 2q^5 - 3q^3 + 3q^1 - 3q^{-1} + 3q^{-3} - 2q^{-5} + q^{-7}$	$-q^7 + 4q^5 - 5q^3 + 5q^1 - 3q^{-1} + q^{-3}$	$-q^3$
7 ₇	(1,-1,0,1,-1,1,0,-1,1)	q	$-q^7 + 4q^5 - 7q^3 + 7q^1 - 6q^{-1} + 2q^{-3}$	$-q^7 + 3q^5 - 4q^3 + 5q^1 - 5q^{-1} + 4q^{-3} - 3q^{-5} + q^{-7}$	$-2q^3 + 6q^1 - 7q^{-1} + 7q^{-3} - 4q^{-5} + q^{-7}$	$-q^{-1}$
8 ₄	(1,-1,0,1,1,-1,0,1,-3)	q^{-1}	$q^7 - 2q^5 + 3q^3 - 4q^1 + 2q^{-1} - q^{-3} - q^{-5} + q^{-7}$	$(q - q^{-1})(q^4 - q^2 + 1 - q^{-2} + q^{-4})$	$-q^7 + q^5 + q^3 - 2q^1 + 4q^{-1} - 3q^{-3} + 2q^{-5} - q^{-7}$	$-q$
8 ₆	(1,-1,1,1,-1,-4)	q^{-3}	$q^7 - 2q^5 + 3q^3 - 3q^1 + q^{-3} - 2q^{-5} + q^{-7}$	$(q - q^{-1})(q^2 + 1 + q^{-2})(q^2 - 1 + q^{-2})$	$-q^7 + 2q^5 - q^3 + 3q^{-1} - 3q^{-3} + 2q^{-5} - q^{-7}$	$-q^3$
8 ₁₁	(1,-1,0,1,-2,1,0,-1,-2)	q^{-3}	$(q^2 - 1 + q^{-2})(q^5 - q^3 - 2q^{-3} + q^{-5})$	$-q^3 + q^{-3}$	$-q^7 + 3q^5 - 3q^3 + 2q^1 + q^{-1} - 2q^{-3} + 2q^{-5} - q^{-7}$	$-q^3$
8 ₁₃	(1,-1,0,1,2,-1,0,1,-2)	q	$-q^7 + 3q^5 - 5q^3 + 4q^1 - 2q^{-1} - q^{-3} + 2q^{-5} - q^{-7}$	$(q - q^{-1})(q^4 - q^2 + 1 - q^{-2} + q^{-4})$	$q^7 - 2q^5 + q^3 + 2q^1 - 4q^{-1} + 5q^{-3} - 3q^{-5} + q^{-7}$	$-q^{-1}$
8 ₁₄	(1,-1,0,1,-1,1,0,-1,-3)	q^{-3}	$q^7 - 3q^5 + 4q^3 - 3q^1 + 2q^{-3} - 3q^{-5} + q^{-7}$	$(q - q^{-1})(q^2 + 1 + q^{-2})(q^2 - 1 + q^{-2})$	$-q^7 + 3q^5 - 2q^3 + 3q^{-1} - 4q^{-3} + 3q^{-5} - q^{-7}$	$-q^3$
8 ₁₅	(1,3,1,1,-1,2)	q^{-7}	$q^3 - 4q^1 + 6q^{-1} - 7q^{-3} + 5q^{-5} - 2q^{-7}$	$(q - q^{-1})(q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6})$	$2q^7 - 5q^5 + 7q^3 - 6q^1 + 4q^{-1} - q^{-3}$	$-q^7$
...						
		A^3	A^1	A^{-1}	A^{-3}	
6 ₁	(1,-1,1,1,-1,-2)	1	$-q^2 + 1 - q^{-2}$	$-q^2 + 2 - q^{-2}$	1	
7 ₂	(-1,1,0,-1,-1,1,0,-1,-3)	-1	$q^2 - 1 + q^{-2}$	$q^2 - 2 + q^{-2}$	$q^2 - 1 + q^{-2}$	
7 ₄	(1,-1,0,1,2,-1,0,1,2)	$q^2 - 1 + q^{-2}$	$q^2 - 1 + q^{-2}$	-1	0	
7 ₆	(-1,1,-1,-1,1,-2)	-1	$2q^2 - 2 + 2q^{-2}$	$-q^4 + 2q^2 - 3 + 2q^{-2} - q^{-4}$	$q^2 - 1 + q^{-2}$	
7 ₇	(1,-1,0,1,-1,1,0,-1,1)	$-q^2 + 2 - q^{-2}$	$q^4 - 2q^2 + 4 - 2q^{-2} + q^{-4}$	$-2q^2 + 2 - 2q^{-2}$	1	
8 ₄	(1,-1,0,1,1,-1,0,1,-3)	$q^2 - 1 + q^{-2}$	$-q^4 + 2q^2 - 2 + 2q^{-2} - q^{-4}$	$-q^4 + q^2 - 2 + q^{-2} - q^{-4}$	$q^2 + q^{-2}$	
8 ₆	(1,-1,1,1,-1,-4)	$q^2 - 1 + q^{-2}$	$-q^4 + 2q^2 - 3 + 2q^{-2} - q^{-4}$	$-q^4 + 2q^2 - 3 + 2q^{-2} - q^{-4}$	$q^2 + q^{-2}$	
8 ₁₁	(1,-1,0,1,-2,1,0,-1,-2)	$q^2 - 1 + q^{-2}$	$-q^4 + 2q^2 - 4 + 2q^{-2} - q^{-4}$	$-q^4 + 3q^2 - 3 + 3q^{-2} - q^{-4}$	$q^2 - 1 + q^{-2}$	
8 ₁₃	(1,-1,0,1,2,-1,0,1,-2)	$-q^2 + 2 - q^{-2}$	$q^4 - 3q^2 + 4 - 3q^{-2} + q^{-4}$	$q^4 - 2q^2 + 4 - 2q^{-2} + q^{-4}$	$-q^2 + 1 - q^{-2}$	
8 ₁₄	(1,-1,0,1,-1,1,0,-1,-3)	$q^2 - 2 + q^{-2}$	$-q^4 + 3q^2 - 4 + 3q^{-2} - q^{-4}$	$-q^4 + 3q^2 - 4 + 3q^{-2} - q^{-4}$	$q^2 - 1 + q^{-2}$	
8 ₁₅	(1,3,1,1,-1,2)	1	$-3q^2 + 2 - 3q^{-2}$	$2q^4 - 3q^2 + 5 - 3q^{-2} + 2q^{-4}$	$q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4}$	
...						

6 5-strand braids

In the five-strand case one deals with traces over seven representations emerging in the decomposition

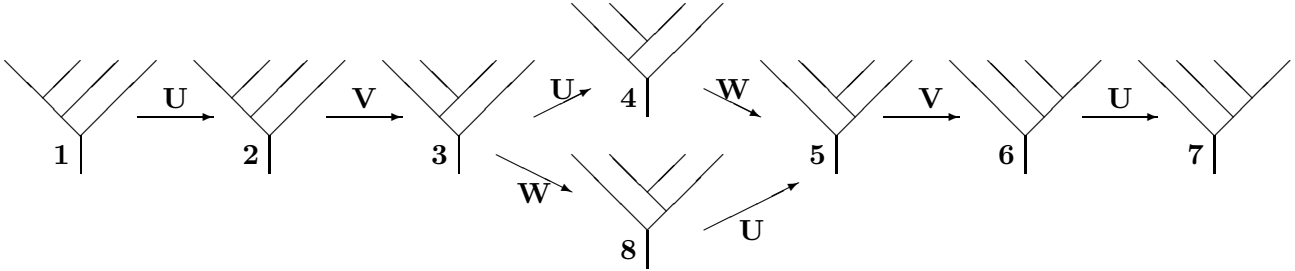
$$[1]^5 = [5] + 4[41] + 5[32] + [311] + [221] + 4[2111] + [11111] \quad (70)$$

The HOMFLY polynomial has the form

$$\begin{aligned} H_{[1]}^{(a_1, b_1, c_1, d_1, \dots)} &= q^{a_1 + b_1 + c_1 + d_1 + \dots} S_5^* + \\ &+ \text{tr} \left[\left(\hat{\mathfrak{R}}_1^{[41]} \right)^{a_1} \underbrace{\hat{U}^{[41]} \left(\hat{\mathfrak{R}}_1^{[41]} \right)^{b_1} \widetilde{\hat{U}^{[41]}}}_{\text{}} \underbrace{\hat{U}^{[41]} \hat{V}^{[41]} \hat{U}^{[41]} \left(\hat{\mathfrak{R}}_1^{[41]} \right)^{c_1} \left(\widetilde{\hat{U}^{[41]} \hat{V}^{[41]} \hat{U}^{[41]}} \right)}_{\text{}} \times \right. \\ &\times \underbrace{\hat{U}^{[41]} \hat{V}^{[41]} \hat{W}^{[41]} \hat{U}^{[41]} \hat{V}^{[41]} \hat{U}^{[41]} \left(\hat{\mathfrak{R}}_1^{[41]} \right)^{d_1} \left(\widetilde{\hat{U}^{[41]} \hat{V}^{[41]} \hat{U}^{[41]} \hat{W}^{[41]} \hat{V}^{[41]} \hat{U}^{[41]}} \right) \dots}_{\text{}} \left. \right] S_{41}^* + \\ &+ \text{tr} \left[\left(\hat{\mathfrak{R}}_1^{[32]} \right)^{a_1} \underbrace{\hat{U}^{[32]} \left(\hat{\mathfrak{R}}_1^{[32]} \right)^{b_1} \widetilde{\hat{U}^{[32]}}}_{\text{}} \underbrace{\hat{U}^{[32]} \hat{V}^{[32]} \hat{U}^{[32]} \left(\hat{\mathfrak{R}}_1^{[32]} \right)^{c_1} \left(\widetilde{\hat{U}^{[32]} \hat{V}^{[32]} \hat{U}^{[32]}} \right)}_{\text{}} \times \right. \\ &\times \underbrace{\hat{U}^{[32]} \hat{V}^{[32]} \hat{W}^{[32]} \hat{U}^{[32]} \hat{V}^{[32]} \hat{U}^{[32]} \left(\hat{\mathfrak{R}}_1^{[32]} \right)^{d_1} \left(\widetilde{\hat{U}^{[32]} \hat{V}^{[32]} \hat{U}^{[32]} \hat{W}^{[32]} \hat{V}^{[32]} \hat{U}^{[32]}} \right) \dots}_{\text{}} \left. \right] S_{32}^* + \\ &+ \left(q \longrightarrow -\frac{1}{q} \right) + \alpha S_{311}^* \end{aligned} \quad (71)$$

where the contributions of $[5]$ is trivial, those of $[41]$ and $[32]$ can be calculated in the $SU(2)$ case, and the contributions of $[221]$, $[2111]$ and $[11111]$ are restored by the symmetry. The only non-trivial contribution that requires the higher group ($SU(3)$) comes from representation $[311]$.

The mixing matrices are described by the following system of the trees:



Note that one can pass from tree 3 to tree 5 both through tree 4 and tree 8. This imposes a consistency condition, which implies that the matrices U and W should commute.

The \mathfrak{R} -matrices in the non-trivial cases are

$$\hat{\mathfrak{R}}^{[41]} = \begin{pmatrix} q & & & \\ & q & & \\ & & q & \\ & & & -\frac{1}{q} \end{pmatrix}, \quad \hat{\mathfrak{R}}^{[32]} = \begin{pmatrix} q & & & \\ & q & & \\ & & -\frac{1}{q} & \\ & & & q \\ & & & & -\frac{1}{q} \end{pmatrix} \quad (72)$$

while the matrices U , V and W in representation [41] can be read off from (60)-(61) and are

$$\hat{U}^{[41]} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & C_2 & S_2 \\ & & -S_2 & C_2 \end{pmatrix}, \quad \hat{V}^{[41]} = \begin{pmatrix} 1 & & & \\ & C_3 & S_3 & \\ & -S_3 & C_3 & \\ & & & 1 \end{pmatrix}, \quad \hat{W}^{[41]} = \begin{pmatrix} C_4 & S_4 & & \\ -S_4 & C_4 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad (73)$$

with

$$C_k = \frac{1}{[k]_q}, \quad S_k = \frac{\sqrt{[k]_q^2 - 1}}{[k]_q} \quad (74)$$

Similarly,

$$\hat{U}^{[32]} = \begin{pmatrix} 1 & & & & \\ & -C_2 & S_2 & & \\ & -S_2 & -C_2 & & \\ & & & -C_2 & S_2 \\ & & & -S_2 & -C_2 \end{pmatrix}, \quad \hat{V}^{[32]} = \begin{pmatrix} -C_3 & S_3 & & & \\ S_3 & C_3 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}, \quad (75)$$

$$\hat{W}^{[32]} = \begin{pmatrix} 1 & & & & \\ & -C_2 & & S_2 & \\ & & -C_2 & & S_2 \\ & S_2 & & C_2 & \\ & & S_2 & & C_2 \end{pmatrix}$$

From these manifest matrices one can check that, indeed,

$$[\hat{U}^{[32]}, \hat{W}^{[32]}] = 0 \quad \text{and} \quad [\hat{U}^{[41]}, \hat{W}^{[41]}] = 0 \quad (76)$$

The Table of HOMFLY polynomials including three 5-strand knots with 8 crossings, 8_1 , 8_3 , 8_{12} and first 3 torus knots with 5 strands

knot	$(a_1, b_1, c_1, d_1 a_2, b_2, c_2, d_2 a_3, b_3, c_3, d_3, \dots)$	A^4	A^2	1	A^{-2}	A^{-4}
8 ₁	$(-2, -1, 0, 0 1, -1, -1, 1 0, 1, -1, 1):$	1	$-q^2 + 1 - q^{-2}$	$-(q - q^{-1})^2$	$-(q - q^{-1})^2$	1
8 ₃	$(-2, -1, 1, 0 1, -1, 1, 1 0, 0, -1, 1):$	1	$-(q - q^{-1})^2$	$-2q^2 + 3 - 2q^{-2}$	$-(q - q^{-1})^2$	1
8 ₁₂	$(-1, 1, -1, 1 -1, 1, -1, 1):$	1	$-2q^2 + 3 - 2q^{-2}$	$q^4 - 3q^2 + 5 - 3q^{-2} + q^{-4}$	$-2q^2 + 3 - 2q^{-2}$	1
...						
[5, 1] unknot	$(1, 1, 1, 1):$	1				
[5, 2] = [2, 5]	$(-1, -1, -1, -1 -1, -1, -1, -1):$				$-[4]_q/[2]_q$	$[6]_q/[2]_q$
[5, 3] = [3, 5]	$(-1, -1, -1, -1 -1, -1, -1, -1 -1, -1, -1, -1):$			$[4]_q/[2]_q$	$[4]_q[2]_q(q^2 - 1 + q^{-2})$	$[7]_q(q^5 - q^4 + q - 1 + q^{-1} - q^{-4} + q^{-5})$
...						
knot	$(a_1, b_1, c_1, d_1 a_2, b_2, c_2, d_2 a_3, b_3, c_3, d_3, \dots)$	S_5^*	S_{41}^* S_{32}^*			
8 ₁	$(-2, -1, 0, 0 1, -1, -1, 1 0, 1, -1, 1):$	q^{-2}	$2q^4 - 4q^2 + 2 - 2q^{-4} + q^{-6}$ $(q - q^{-1})\frac{q^{10} - 2q^8 + 3q^6 - 4q^4 + 2q^2 - 1}{q^3}$			
8 ₃	$(-2, -1, 1, 0 1, -1, 1, 1 0, 0, -1, 1):$	1	$q^6 - q^4 - q^2 + 1 - q^{-2} - q^{-4} + q^{-6}$ $(q^2 - 1 + q^{-2})(q - q^{-1})^2$			
8 ₁₂	$(-1, 1, -1, 1 -1, 1, -1, 1):$	1	$2q^4 - 6q^2 + 7 - 6q^{-2} + 2q^{-4}$ $[4]_q/[2]_q(q - q^{-1})^2(q^2 - 1 + q^{-2})^2$			
...						
[5, 1] unknot	$(1, 1, 1, 1):$	q^4	$-q^2$ 0			
[5, 2] = [2, 5]	$(-1, -1, -1, -1 -1, -1, -1, -1 -1, -1, -1, -1):$	q^{-8}	$-q^{-4}$ 0			
[5, 3] = [3, 5]	$(-1, -1, -1, -1 -1, -1, -1, -1 -1, -1, -1, -1):$	q^{-12}	q^{-6} 0			
...						
knot		S_{311}^*	S_{221}^*	S_{2111}^*	S_{11111}^*	
8 ₁	$-2q^6 + 4q^4 - 4q^2 + 5 - 4q^{-2} + 4q^{-4} - 2q^{-6}$		$(q - q^{-1})(q^3 - 2q + 4q^{-1} - 3q^{-3} + 2q^{-5} - q^{-7})$	$q^6 - 2q^4 + 2 - 4q^{-2} + 2q^{-4}$	q^2	
8 ₃	$-2q^6 + 3q^4 - q^2 + 1 - q^{-2} + 3q^{-4} - 2q^{-6}$		$(q^2 - 1 + q^{-2})(q - q^{-1})^2$	$q^6 - q^4 - q^2 + 1 - q^{-2} - q^{-4} + q^{-6}$	1	
8 ₁₂	$q^8 - 6q^6 + 11q^4 - 12q^2 + 13 - 12q^{-2} + 11q^{-4} - 6q^{-6} + q^{-8}$		$[4]_q/[2]_q(q - q^{-1})^2(q^2 - 1 + q^{-2})^2$	$2q^4 - 6q^2 + 7 - 6q^{-2} + 2q^{-4}$	1	
...						
[5, 1] unknot	1		0	$-q^{-2}$	q^{-4}	
[5, 2] = [2, 5]	1		0	$-q^4$	q^8	
[5, 3] = [3, 5]	1		0	q^6	q^{12}	
...						

7 Summary and comments

The goal of this paper was to present an expansion of the HOMFLY polynomials of knots into characters (the Schur functions)

$$H_R^K = \sum_{Q \vdash m|R|} h_R^Q S_Q^* \quad (77)$$

A priori, such an expansion is ambiguous for $m > 3$, and we suggest to define it through the braid realization \mathcal{B} of the knot \mathcal{K} , which contains m strands, and present explicit formulas for h_R^Q for $R = [1]$ and $m = 2, 3, 4$. In the case of $m = 5$ one of the coefficients, $h_{[1]}^{[311]}$ is left theoretically undetermined in the present paper, and we read off its values for 5-strand knots from the known answers for the HOFMLY polynomials.

The answer for a fixed braiding representation of the knot is completely expressed through the known eigenvalues of the quantum \mathcal{R} -matrices and through the mixing \mathcal{U} -matrices that relates \mathcal{R} -matrices acting on different strands of the braid. The matrix elements of these mixing matrices are the Racah coefficients, they turn out to look very simple and exhibit vast universality properties.

More concretely, for the m -strand braid with the representation R running through each strand, we worked in the basis of irreps at each transverse slice of the braid, $R^{\otimes m} \rightarrow \oplus Q$. There were a few crucial ingredients in the construction.

- Typically, there emerge several similar irreps in this expansion. They are labeled by the path leading to the concrete irrep through the fusion tree T . Since we considered here only the trees with the vertices where at least one of the edges carries $R = [1]$, we can describe the path through the tree only with the sequence of non-trivial entries: $V_T(R_1 \rightarrow R_2 \rightarrow R_3 \dots)$.
- Thus, typically there is a set of similar irreps for the given tree. These sets for different trees are connected by the mixing matrices. We require the vectors in the irrep to be normalized. Then, the mixing matrices are orthogonal.
- If there are N different similar irreps, the mixing matrix is of the size $N \times N$. However, we demonstrated that in the considered cases it is really 2×2 matrix, because of the specific structure of mixing. It is illustrated in the Table, where the representations mixing are marked by the same letter:

paths	U	V	W
level	1	2	3
mixing reps	[2]&[11]	[3]&[21]	[4]&[31] [31]&[22]
$V_T([2] \rightarrow [3] \rightarrow [4] \rightarrow [41])$			z
$V_T([2] \rightarrow [3] \rightarrow [31] \rightarrow [41])$		y	z
$V_T([2] \rightarrow [21] \rightarrow [31] \rightarrow [41])$	x	y	
$V_T([11] \rightarrow [21] \rightarrow [31] \rightarrow [41])$	x		
$V_T([2] \rightarrow [3] \rightarrow [31] \rightarrow [32])$		c	
$V_T([2] \rightarrow [21] \rightarrow [31] \rightarrow [32])$	a	c	e
$V_T([11] \rightarrow [21] \rightarrow [31] \rightarrow [32])$	a		d
$V_T([2] \rightarrow [21] \rightarrow [22] \rightarrow [32])$	b		e
$V_T([11] \rightarrow [21] \rightarrow [22] \rightarrow [32])$	b		d

One can see that only the paths that differ from each other exactly in one item can mix, which effectively leads to 2×2 matrices.

- Actually, the mixing matrices $\mathcal{U}_{\mu\nu}$ have their own hierarchical structure, expressed as a relation between two decorated routed trees, decorated by the representation in the decomposition

$$R^{\otimes m} = \sum_Q \mathcal{M}_Q \otimes Q \quad (78)$$

\mathcal{M}_Q is actually labeled by a 3-valent tree, and \mathcal{U} matrix is decomposed into a product of elementary constituents (U, V, W, \dots) , realizing elementary steps, that is, flips of a single edge in the routed 3-valent tree.

- In order to calculate the knot invariant one also needs to know the eigenvalues of \mathcal{R} -matrix. Then, one chooses the \mathcal{R} -matrix acting at the first two strands diagonal and expresses \mathcal{R} -matrices acting at other strands through diagonal \mathcal{R} -matrix rotated with the mixing matrices. It immediately gives the knot polynomial.
- Unfortunately, the coefficients of expansion (77) h_R^Q are not knot invariant. For instance, the HOMFLY polynomial for toric knot $T[n, m]$ is the same as that for toric knot $T[m, n]$, but the numbers of strands and, therefore, expansions (77) are different for them. However, it looks plausible that while the full knot-invariance is lost in transfer to h_R^Q , some rich knot-covariance w.r.t. switches between different braid realizations can be finally found for the character expansion (77).

Among the advantages of the character expansion (77) we emphasize the following:

- Its coefficients do not depend on A , which allows one to calculate them for small rank groups $SU_q(l)$, with needed l depending only on R and m , still knot invariants are obtained at once for all the values of the rank of the *gauge* group $SU(N)$. Moreover, the answers appear directly expressed through $A = q^N$ as required for the HOMFLY polynomials (what is not at all automatic for other calculations in Chern-Simons theory). This gives a good technical alternative to the usually explored in calculating skein relations.
- There are also two more advantages of more theoretical nature (see [7]). First of all, the character expansion can be naturally extended to new set of variables: one can consider the Schur functions at arbitrary points. This opens a way to dealing with new set of questions related, e.g., to integrability. Second, one can effectively use this expansion in order to immediately construct (quantum) A -polynomials.

We plan to develop this formalism and its applications in the forthcoming publications.

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